Calibrations for the Steiner Problem in a Covering Space Setting

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Transport problems in Zurich

Curvature, Cycles, and Cohomology

Universität Regensburg

Let $S = \{p_1, \dots, p_m\}$ be a finite set in $\mathbb{R}^2$.

Problem (Steiner)

Find a connected set Γ in $\mathbb{R}^2$ such that $S \subset \Gamma$ and the length of Γ is minimal.

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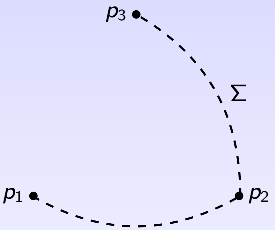
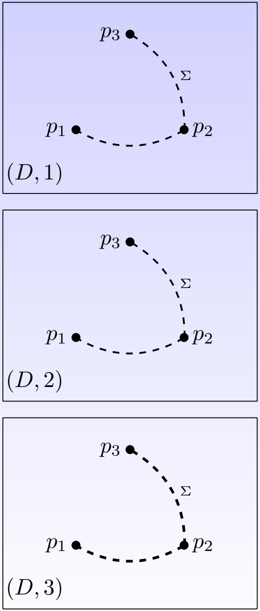
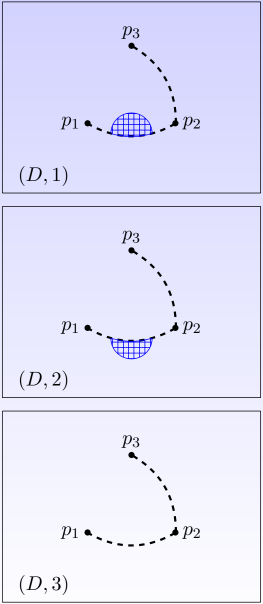
Idea: Minimizing the total variation of suitable defined BV–functions in a m–sheeted covering space is equivalent to minimize the length of a network that connects the m points of S in the plane.
(Amato-Bellettini-Paolini)


p_3
Σ
p_1
p_2

Construction of the covering space Y of $M := \mathbb{R}^2 \setminus S$



A closer look at the topology of the covering space



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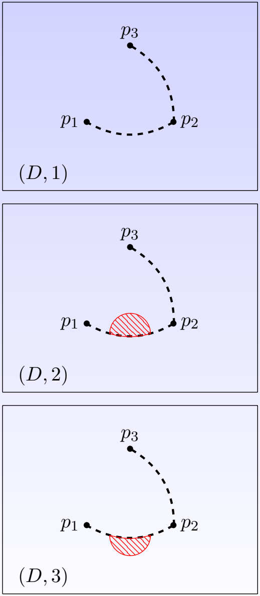

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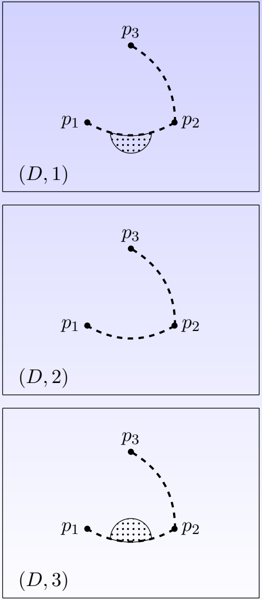

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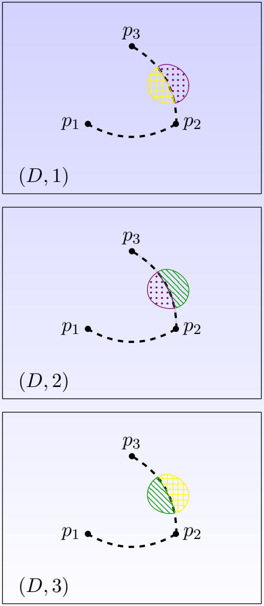
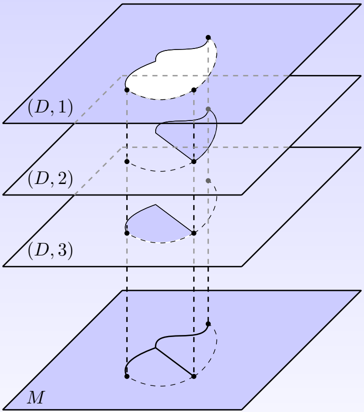
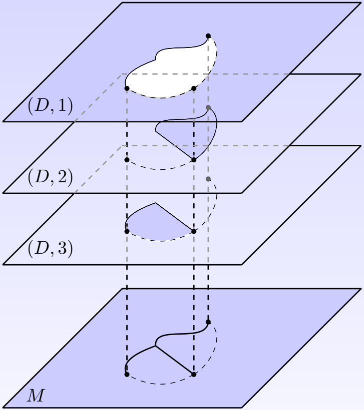




We say that a function $u \in BV(Y, \{0,1\})$ is *constrained* if

$$\sum_{p(y)=x} u(y) = 1\,,$$

for every $x \in M = \mathbb{R}^2 \setminus S$.
Moreover we add a suitable boundary condition.



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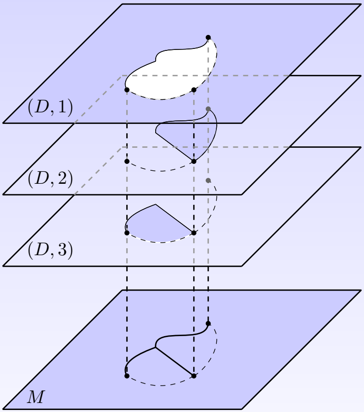
Proposition

Given $u \in BV_{constr}(Y)$ there hold:

- $|Du|(Y) = 2\mathcal{H}^1(p(J_u))$,



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Proposition

Given $u \in BV_{constr}(Y)$ there hold:

- $|Du|(Y) = 2\mathcal{H}^1(p(J_u))$,
- *the set S is contained in a connected component of $\overline{p(J_u)}$.*

Problem (∗)

Find a function $u \in BV_{constr}(Y)$ such that its total variation is minimal.

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Theorem

Problem (∗) and the Steiner problem are equivalent.
To be more precise the solution of the Steiner problem is the connected network $\overline{p(J_{u_{\min}})}$, where $u_{\min} \in BV_{constr}(Y)$ is a solution of Problem (∗).

Given a vector field $\Phi : Y \to \mathbb{R}^2$, we denote with $\Phi_i : M \to \mathbb{R}^2$ the pushforward (according to the projection $p : Y \to M$) of the restriction of Φ on the i-th sheet of the covering Y.

Definition

Let $u \in BV_{\mathrm{constr}}(Y)$, a calibration for u is a vector field $\Phi : Y \to \mathbb{R}^2$ such that

1. $div\Phi = 0$;
2. $|\Phi_i - \Phi_j| \leq 2$ *for every* $i, j = 1, \ldots m$;
3. $\int_Y \Phi \cdot Du = |Du|(Y)$.

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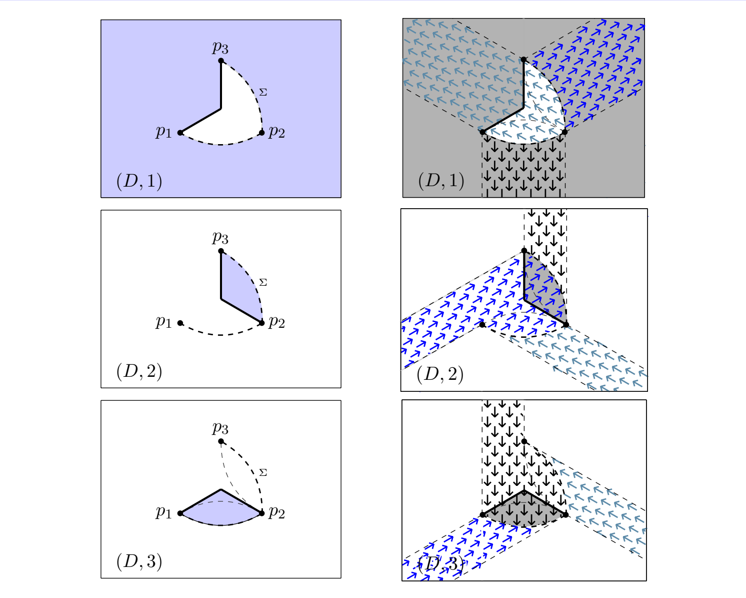
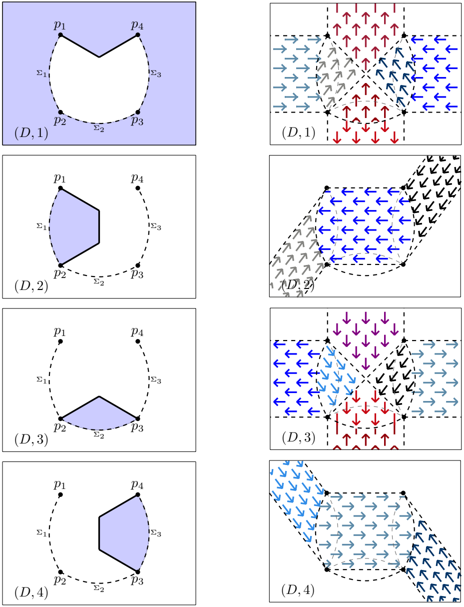
Theorem

If $\Phi : Y \to \mathbb{R}^2$ is a calibration for $u \in BV_{\mathrm{constr}}(Y)$, then u is a minimizer for Problem $(*)$.

Example: calibration for 3 points



Example: calibration for 4 points



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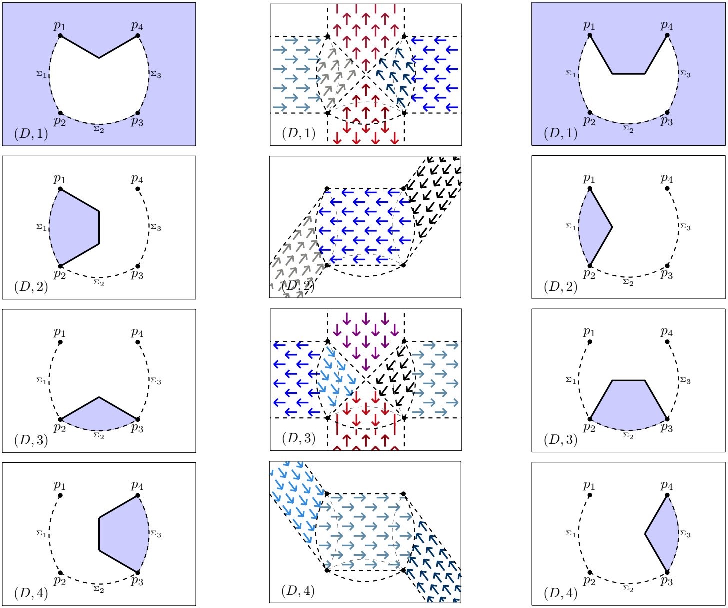

Consider a candidate minimizer $u \in BV_{\mathrm{constr}}(Y)$ and a competitor $v \in BV_{\mathrm{constr}}(Y)$.

We have

$$
\begin{aligned}
&|Du|(Y) \overset{(3)}{=} \int_Y \Phi \cdot Du \overset{(1)}{=} \int_Y \Phi \cdot Dv \\
=&\sum_{j=1}^{m} \int_D \Phi_j \cdot Dv_j = \sum_{j=1}^{m} \int_{J_{v_j}} \Phi_j (v_j^+ - v_j^-) \cdot \nu \, d\mathcal{H}^1 \\
\leq& \int_{p(J_v)} \Big| \sum_{j=1}^{m} \phi_j (v_j^+ - v_j^-) \Big| \, d\mathcal{H}^1 \overset{(2)}{\leq} 2\mathcal{H}^1(p(J_v)) \\
=&|Dv|(Y)
\end{aligned}
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$$\leq \int_{p(J_v)} \left| \sum_{j=1}^{m} \phi_j (v_j^+ - v_j^-) \right| d\mathcal{H}^1 \overset{(2)}{\leq} 2\mathcal{H}^1(p(J_v))$$

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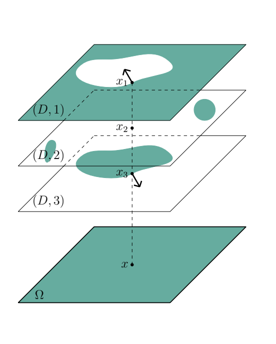

The last inequality holds because

$$(v_j^+ - v_j^-)(x) = \begin{cases} +1 & \text{for } j = j_1 \\ -1 & \text{for } j = j_2 \\ 0 & \text{for all } j \neq j_1, j_2 \end{cases}$$

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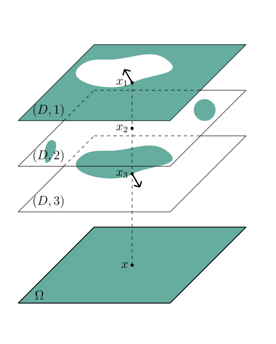

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Moreover fixed a competitor $v \in BV_{constr}(Y)$, property (2) is necessary only for (i,j) such that

$$\mathcal{H}^1(J_{v_i} \cap J_{v_j}) > 0 \,.$$

Definition

Let $\mathcal{J} \subset \{1, \dots, m\} \times \{1, \dots, m\}$, we define

$$\mathcal{F}(\mathcal{J}) = \{u \in BV_{constr}(Y) : \mathcal{H}^1(J_{u_i} \cap J_{u_j}) = 0, \text{ for every } (i,j) \in \mathcal{J}\}.$$

Definition

A calibration for u in $\mathcal{F}(\mathcal{J})$ is a vector field $\Phi : Y \to \mathbb{R}^2$ such that

1. $div\Phi = 0$;
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If there exists a calibration Φ_i for u_i in $\mathcal{F}(\mathcal{J}^i)$, then u_i is a minimizer in $\mathcal{F}(\mathcal{J}^i)$. Suppose moreover that there exist $\mathcal{J}_1, \dots, \mathcal{J}_N$ such that $BV_{constr}(Y) = \bigcup_{i=1}^N \mathcal{F}(\mathcal{J}^i)$ then the solution of Problem $(*)$ is the u_i with less energy among the minimizers in $\mathcal{F}(\mathcal{J}^i)$.

A useful remark

We restrict to the case in which the points are located on the boundary of a convex set, moreover we consider as possible competitors the BV_{constr} functions associated to connected networks without loops.

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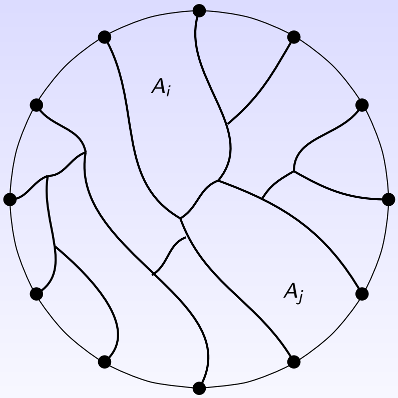

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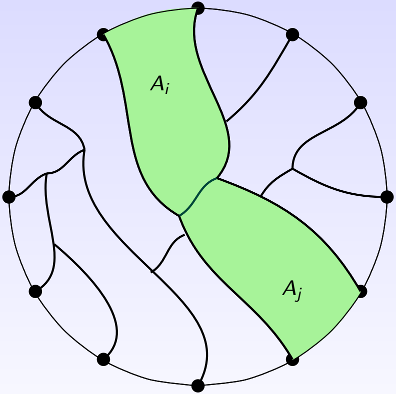

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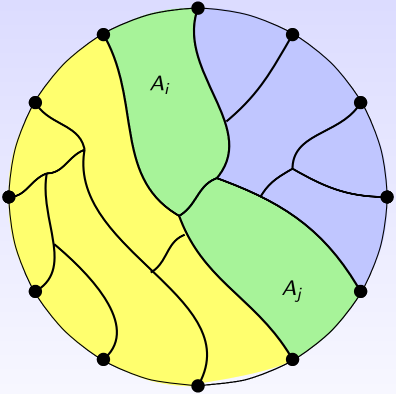
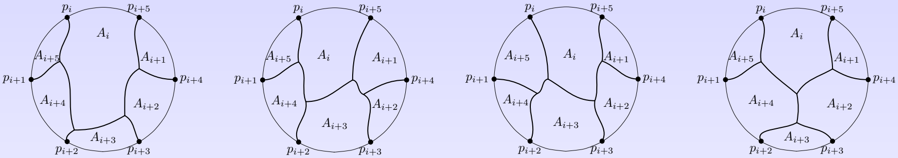


Example: the families $\mathcal{F}(\mathcal{J})$ for 6 points

Consider 6 points that lie at the vertices of a regular hexagon. Thanks to the previous remark we find that the competitors can be divides in the following families:



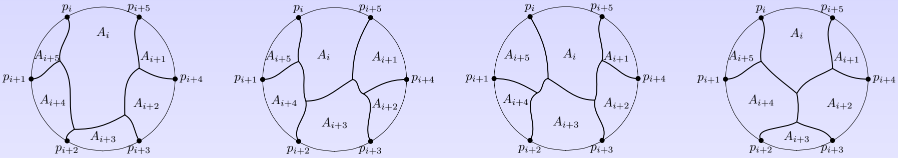
For example in the first picture

$$\partial A_{i+1} \cap \partial A_{i+3} = \emptyset,\ \partial A_{i+1} \cap \partial A_{i+4} = \emptyset,$$
$$\partial A_{i+1} \cap \partial A_{i+5} = \emptyset,\ \partial A_{i+2} \cap \partial A_{i+4} = \emptyset,$$
$$\partial A_{i+2} \cap \partial A_{i+5} = \emptyset,\ \partial A_{i+3} \cap \partial A_{i+5} = \emptyset,$$

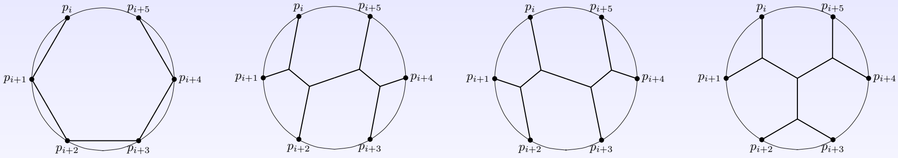
with $i \in \{1, 2, 3, 4, 5, 6\}$ and the indices cyclically identified modulus 6.

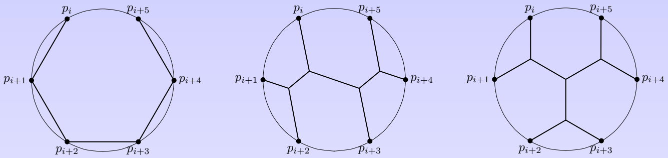
Example: 6 points

In each family



we prove, exhibiting a calibration, that the minimizers are the following:



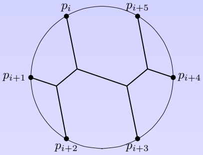


We get

$$\Phi_1^1 = (0,0)\,, \qquad \Phi_2^1 = (\sqrt{3},1)\,, \qquad \Phi_3^1 = (\sqrt{3},-1)\,,$$
$$\Phi_4^1 = (0,-2)\,, \qquad \Phi_5^1 = (-\sqrt{3},-1)\,, \qquad \Phi_6^1 = (-\sqrt{3},1)\,,$$

$$\Phi_1^2 = (0,0)\,, \qquad \Phi_2^2 = \left(\frac{3\sqrt{3}}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)\,, \qquad \Phi_3^2 = \left(\frac{2\sqrt{3}}{\sqrt{7}}, -\frac{4}{\sqrt{7}}\right)\,,$$
$$\Phi_4^2 = \left(-\frac{\sqrt{3}}{\sqrt{7}}, -\frac{5}{\sqrt{7}}\right)\,, \qquad \Phi_5^2 = \left(-\frac{4\sqrt{3}}{\sqrt{7}}, -\frac{6}{\sqrt{7}}\right)\,, \qquad \Phi_6^2 = \left(-\frac{3\sqrt{3}}{7}, -\frac{1}{\sqrt{7}}\right)\,,$$

$$\Phi_1^3 = (0,0)\,, \qquad \Phi_2^3 = (2,0)\,, \qquad \Phi_3^3 = (1,-\sqrt{3})\,,$$
$$\Phi_4^3 = (0,-2\sqrt{3})\,, \qquad \Phi_5^3 = (-1,-\sqrt{3})\,, \qquad \Phi_6^3 = (-2,0)\,.$$



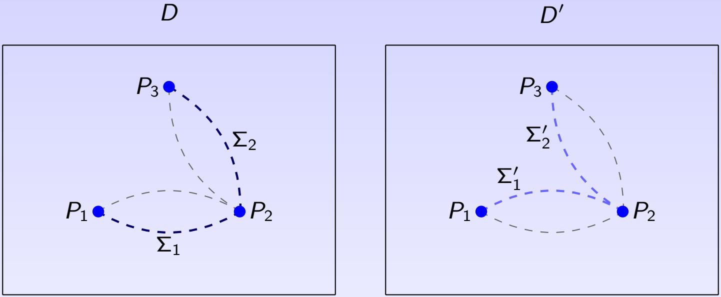
Rotating the points of S, the network and consequently changing the associate function by a rotation that fix the origin where the matrix of the transformation is

$$\begin{bmatrix} \frac{3\sqrt{3}}{2\sqrt{7}} & -\frac{1}{2\sqrt{7}} \\ \frac{1}{2\sqrt{7}} & \frac{3\sqrt{3}}{2\sqrt{7}} \end{bmatrix}$$

we obtain the calibration

$$\begin{array}{lll} \widetilde{\Phi}_1^{10} = (0,0)\,, & \widetilde{\Phi}_2^{10} = (2,0)\,, & \widetilde{\Phi}_3^{10} = (1,-\sqrt{3})\,, \\ \widetilde{\Phi}_4^{10} = (-1,-\sqrt{3})\,, & \widetilde{\Phi}_5^{10} = (-3,-\sqrt{3})\,, & \widetilde{\Phi}_6^{10} = (-2,0)\,. \end{array}$$

It is more easy see the analogy between $\widetilde{\Phi}^2$ and Φ^1, Φ^3.

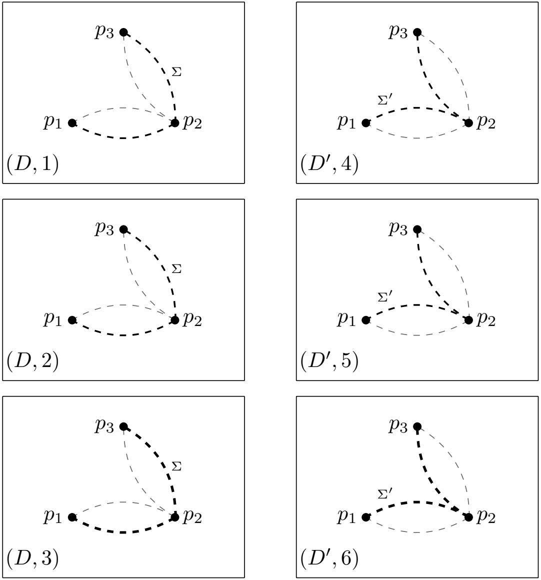


We call $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma' = \Sigma_1' \cup \Sigma_2'$ admissible cuts if

- Σ_i, Σ_i' are Lipschitz simple curves starting at p_i and ending at p_{i+1};
- $\Sigma \cap \Sigma' = S$.

Let I_i be the open and bounded set enclosed in Σ_i and Σ_i' and $O = \mathbb{R}^2 \setminus \bigcup_{i=1}^{m-1} \bar{I}_i$.
We call $D = \mathbb{R}^2 \setminus \Sigma$ and $D' = \mathbb{R}^2 \setminus \Sigma'$.

Construction of the covering space Y of $M := \mathbb{R}^2 \setminus S$

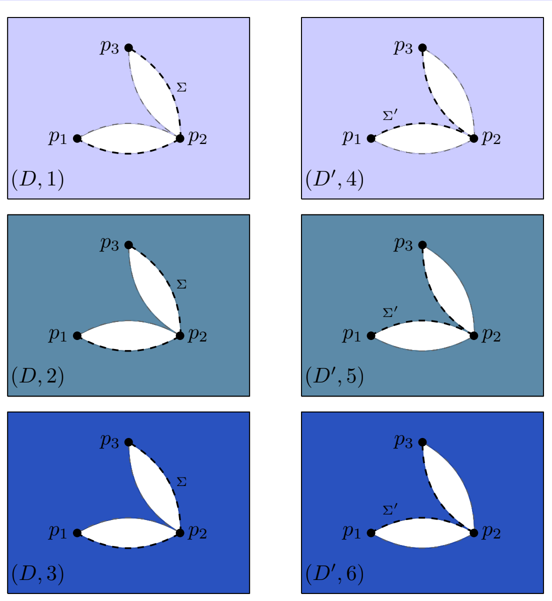


We call $\mathbf{D} := (\cup_j (D, j)) \bigcup (\cup_{j'} (D', j'))$.

Given
$(x, j) \in (D, j)$ with $j \in \{1, \ldots, m\}$ and $(x', j') \in (D', j')$ with $j' \in \{m+1, \ldots, 2m\}$, we define the equivalence relation $\sim$ in $\mathbf{D}$ as $(x, j) \sim (x', j')$ if and only if one of the following condition holds:

$$\begin{cases} j \equiv j' \pmod{m}, & x = x' \in O\,, \\ j \equiv j' - i \pmod{m}, & x = x' \in I_i, \ \ i = 1, \ldots, m-1\,. \end{cases}$$

Construction of the covering space Y of $M := \mathbb{R}^2 \setminus S$



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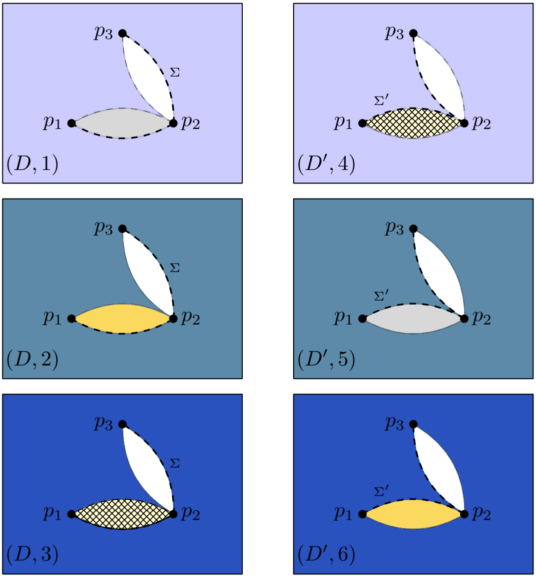
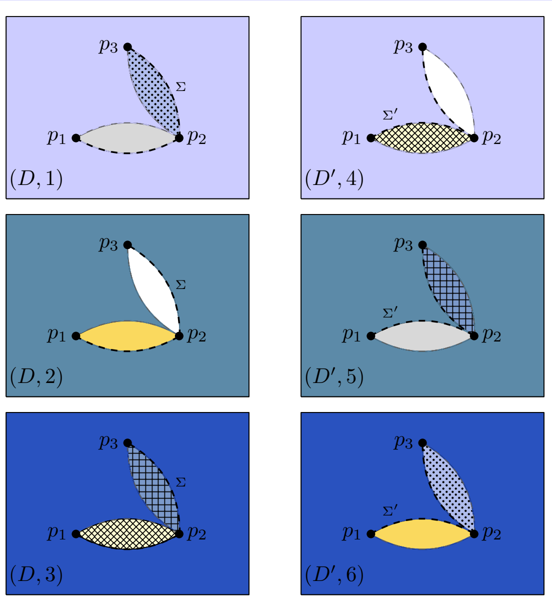


Construction of the covering space Y of $M := \mathbb{R}^2 \setminus S$



Construction of the covering space Y of $M = \mathbb{R}^2 \setminus \{p_1, \ldots, p_m\}$.

We define Y to be the topological quotient space induced by $\sim$, i.e.

$$Y := \mathbf{D} \big/ \sim \; .$$

We denote by $\tilde{\pi} : \mathbf{D} \to Y$ the projection induced by the equivalence relation, by π the projection from $\mathbf{D}$ to the base space M and by $p : Y \to M$ the map that makes the following diagram commute:

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{\tilde{\pi}} & Y \\ & \searrow^{\pi} & \downarrow p \\ & & M \end{array}$$

Proposition

The map $p : Y \to M$ exists and the couple (Y, p) is a covering space of M.

The space Y is a manifold.

Given a Borel set $E \subset Y_\Sigma$ we define

$$\mu(E) := \sum_{j=1}^{m} \psi_{j\sharp}\mathcal{L}^2(E \cap \tilde{\pi}((D,j)))\,.$$

Moreover we set $L^1(Y_\Sigma) := L^1(Y_\Sigma;\mathbb{R};\mu)$ and analogously, $L^1_{loc}(Y_\Sigma) := L^1_{loc}(Y_\Sigma;\mathbb{R};\mu)$. We also define the distributional gradient of an function $u \in L^1(Y_\Sigma) := L^1(Y_\Sigma;\mathbb{R};\mu)$ as the linear map

$$Du(\psi) = -\int_{Y_\Sigma} u\, div\psi\, d\mu$$

for $\psi \in C^1_c(Y_\Sigma,\mathbb{R}^2)$.

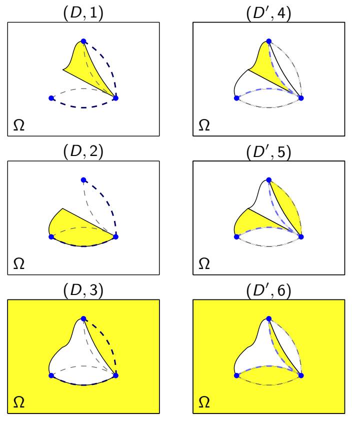
Definition

Given $u \in L^1(Y_\Sigma)$ we say that $u \in BV(Y_\Sigma)$ if Du is a Radon measure with bounded total variation.

The total variation of $u \in L^1(Y_\Sigma)$ denoted with $|Du|$ is defined as

$$|Du|(E) = \sup\left\{\int_E u\, div\psi\, d\mu : \psi \in C^1_c(E,\mathbb{R}^2), \|\psi\| \leq 1\right\}$$

for every open set $E \subset Y_\Sigma$. Moreover let $J_u \subset Y_\Sigma$ be the jump set of u .



Let E be a Borel set of Y_Σ and $u \in BV(Y_\Sigma)$. Then setting $E_j := E \cap \tilde{\pi}((D,j))$ and $E_{j'} := E \cap \tilde{\pi}((\Sigma \setminus S, j'))$ we have $|Du|(E) =$

$$\sum_{j=1}^{m}(\psi_{j\#}|Dv_j(u)|)(E_j) + \sum_{j'=m+1}^{2m}(\psi_{j\#}|Dv_{j'}(u)|)(E_{j'}) .$$

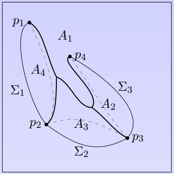
Theorem (Regularity)

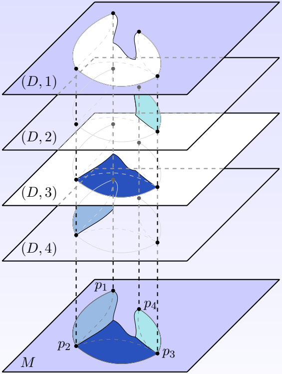
Given $u_{min} \in BV_{constr}(Y_\Sigma)$ let us suppose that that it is a minimizer for Problem $(*)$ Then it holds

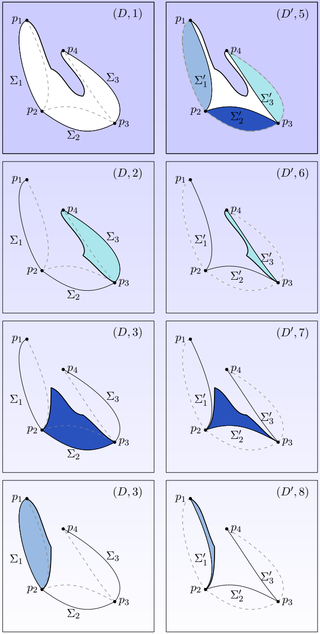
$$\mathcal{H}^1(\overline{p(J_{u_{min}})} \setminus p(J_{u_{min}})) = 0 \quad (1)$$

and the set $\overline{p(J_{u_{min}})}$ is a finite union of segments meeting at triple junctions with angles of 120 degrees.

Canonic construction of a function from a network







Definition (Approximately regular vector fields on $\mathbb{R}^n$)

Given $A \subset \mathbb{R}^n$, a vectorfield $\Phi : A \to \mathbb{R}^n$ is approximately regular if it is bounded and for every Lipschitz hypersurface M in $\mathbb{R}^n$ there holds

$$\lim_{r \to 0} \int_{B_r(x_0) \cap A} |(\Phi(x) - \Phi(x_0)) \cdot \nu_M(x_0)| \, dx = 0 \tag{2}$$

for $\mathcal{H}^{n-1}$-a.e. $x_0 \in M \cap A$.

If Φ admits traces on M, denoted by Φ^+ and Φ^-, then condition (**??**) is equivalent to the following one:

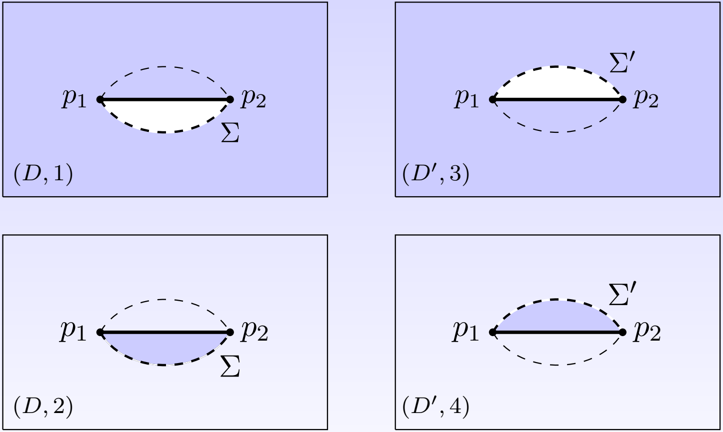
$$\Phi^+(x) \cdot \nu_M(x) = \Phi^-(x) \cdot \nu_M(x) = \Phi(x) \cdot \nu_M(x), \tag{3}$$

for $\mathcal{H}^{n-1}$-a.e. $x_0 \in M \cap A$.

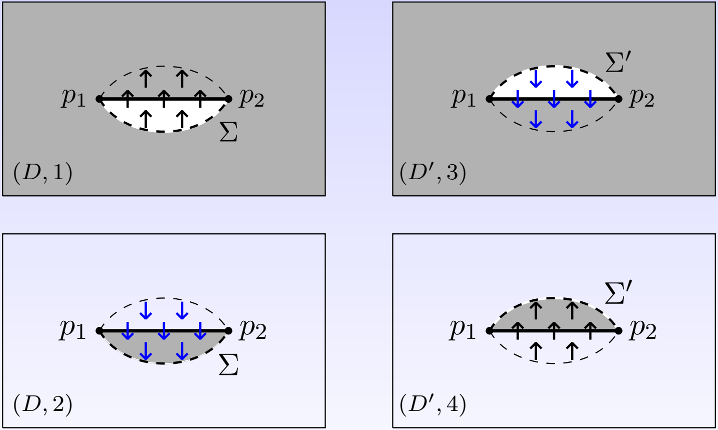
Definition (Approximately regular vector fields on Y_Σ)

Given $\Phi : Y_\Sigma \to \mathbb{R}^2$, we say that it is approximately regular in Y_Σ if $v_j(\Phi)$ and $v_{j'}(\Phi)$ are approximately regular for every $j = 1, \ldots, m$ and $j' = m+1, \ldots, 2m$.

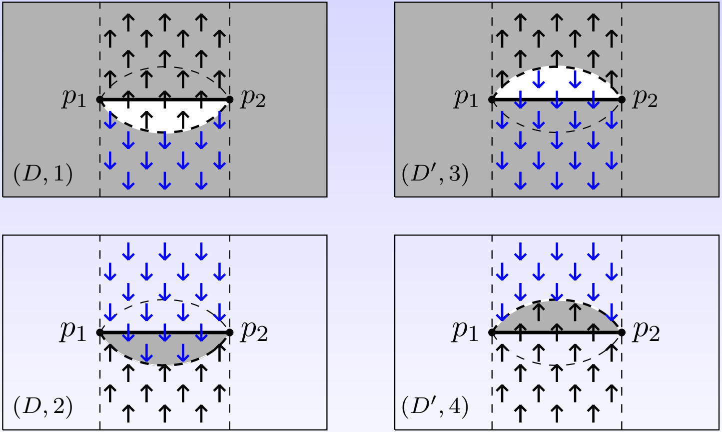
- **Calibration for the segment**



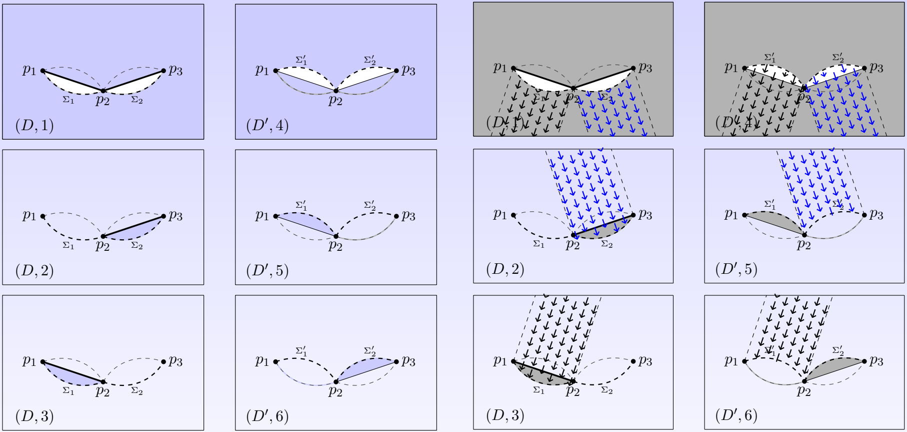
- **Calibration for the segment**



- **Calibration for the segment**



Example


(D, 1)
(D′, 4)
(D, 1)
(D′, 4)
(D, 2)
(D′, 5)
(D, 2)
(D′, 5)
(D, 3)
(D′, 6)
(D, 3)
(D′, 6)
p_1
p_2
p_3
Σ_1
Σ_2
Σ'_1
Σ'_2

Example