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# From linear potential theory to the inverse mean curvature flow

## Monotonicity formulas

with L. Benatti, M. Fogagnolo, L. Mazzieri, and M. Pozzetta

*Geometric Measure Theory on Metric Spaces with Applications to Physics and Geometry*

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**Alessandra Pluda**
alessandra.pluda@unipi.it

Result

Regularity/geometric interpretation of the level sets of $p$-harmonic functions.

Aims

- link monotonicity formulas along the inverse mean curvature flow to monotonicity formulas in (non-)linear potential theory,
- justify formal computations,
- geometric inequalities.

**Setting**

$(M^n, g)$ complete noncompact Riemannian $n$-manifold, $n \geq 3$.

- **Setting 1**: nonnegative Ricci curvature $\mathrm{Ric} \geq 0$;
- **Setting 2**: Riemannian $3$-manifolds, nonnegative scalar curvature $\mathrm{R} \geq 0$.

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# 1 | IMCF and harmonic functions

## Definition - Inverse Mean Curvature Flow

Let $N$ be an hypersurface in $(M^n, g)$ a $n$-Riemannian smooth manifold, and $F_0 : N \to M$ a smooth immersion. A classical solution to the Inverse Mean Curvature flow is a family of smooth immersions $F : [0,T) \times N \to M$ satisfying

$$\begin{cases} \partial_t F(p,t) = \frac{\nu(F(t,p))}{\mathrm{H}(F(t,p))} \\ F(0,p) = F_0(p) \end{cases} \tag{IMCF}$$

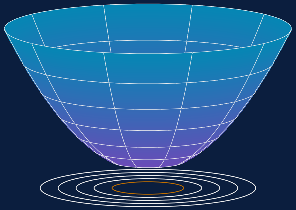
where $\mathrm{H}$ is the mean curvature and $\nu$ the outward normal vector.

## Expanding spheres

Let $N = \mathbb{S}^{n-1}$, $M = \mathbb{R}^n$, $F_0(N) = \mathbb{S}^{n-1}_{R_0}$. Then, IMCF reduces to the following ODE

$$\begin{cases} R'(t) = \frac{1}{n-1} R(t) \\ R(0) = R_0 \end{cases}$$

thus, $R(t) = R_0 \mathrm{e}^{\frac{1}{(n-1)t}}$.



**Setting 1**: $(M,g)$ complete, noncompact Riemannian 3-manifold with nonnegative Ricci curvature $\mathrm{Ric}\geq 0$, $\Sigma$ hypersurface in $M$.

Let $\Sigma_t$ be the IMCF from $\Sigma$. We compute the evolution of the area, the volume and the mean curvature:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Sigma_t| = |\Sigma_t|, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(\Sigma_t) = \int_{\Sigma_t}\frac{1}{\mathrm{H}}\,\mathrm{d}\sigma,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H} = -\Delta\left(\frac{1}{\mathrm{H}}\right) - \frac{|\mathrm{h}|^2}{\mathrm{H}} - \frac{\mathrm{Ric}(\nu,\nu)}{\mathrm{H}}.$$

Then, the evolution of the Willmore functional reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\mathrm{H}^2\,\mathrm{d}\sigma = \int_{\Sigma_t} -2\mathrm{H}\Delta(1/\mathrm{H}) - 2\mathrm{H}\Delta(1/\mathrm{H}) - 2|\mathrm{h}|^2 - 2\mathrm{Ric}(\nu,\nu) + \mathrm{H}^2\,\mathrm{d}\sigma$$

$$\overset{2|\mathrm{h}|^2-\mathrm{H}^2=2|\mathring{\mathrm{h}}|^2}{=} \int_{\Sigma_t} -2\frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2} - 2|\mathring{\mathrm{h}}|^2 - 2\underbrace{\mathrm{Ric}(\nu,\nu)}_{\geq 0}\,\mathrm{d}\sigma \leq 0.$$

where we used $0 = \int \mathrm{div}(\mathrm{H}\nabla(1/\mathrm{H}))\,\mathrm{d}\sigma = \int \mathrm{H}\Delta(1/\mathrm{H}) - |\nabla^\top \mathrm{H}|^2/\mathrm{H}^2\,\mathrm{d}\sigma$.

$\int_{\Sigma_t}\mathrm{H}^2\,\mathrm{d}\sigma$ is monotone non-increasing.

**Setting 2**: $(M,g)$ complete, noncompact Riemannian 3-manifold with nonnegative scalar curvature $\mathrm{R}\geq 0$, $\Sigma$ hypersurface in $M$.

Consider the Hawking mass

$$\mathfrak{m}_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}}\left(1-\frac{1}{16\pi}\int_\Sigma \mathrm{H}^2\,\mathrm{d}\sigma\right).$$

Let $\Sigma_t$ be the IMCF from $\Sigma$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H(\Sigma_t) = \frac{1}{(16\pi)^{3/2}}\left[\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{|\Sigma_t|}\left(16\pi - \int_{\Sigma_t}\mathrm{H}^2\,\mathrm{d}\sigma\right) - \sqrt{|\Sigma_t|}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\mathrm{H}^2\,\mathrm{d}\sigma\right)\right]$$

$$= \frac{1}{16\pi}\sqrt{\frac{|\Sigma_t|}{16\pi}}\left(8\pi - \int_{\Sigma_t}\frac{\mathrm{H}^2}{2}\,\mathrm{d}\sigma + \int_{\Sigma_t} 2\frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2} + 2|\mathring{\mathrm{h}}|^2 + 2\mathrm{Ric}(\nu,\nu)\,\mathrm{d}\sigma\right)$$

$$\overset{2\mathrm{Ric}(\nu,\nu)-\mathrm{H}^2/2=\mathrm{R}-\mathrm{R}^\top-|\mathring{\mathrm{h}}|^2}{=} \frac{1}{16\pi}\sqrt{\frac{|\Sigma_t|}{16\pi}}\left(\underbrace{8\pi - \int_{\Sigma_t}\mathrm{R}^\top\,\mathrm{d}\sigma}_{\geq 0 \text{ Gauss-Bonnet}} + \int_{\Sigma_t}|\mathring{\mathrm{h}}|^2 + \mathrm{R} + \frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2}\,\mathrm{d}\sigma\right) \geq 0.$$

$\mathfrak{m}_H(\Sigma_t)$ is monotone non-decreasing.

Assume that the flow is given by the level set of a function $u: M \setminus \Omega \to \mathbb{R}$ such that $\{x : u(x) = t\} = F(t, \cdot)$. Then, for $p \in N$ we have

$$1 = \frac{\partial}{\partial t} t = \frac{\partial}{\partial t} u(F(t,p)) = \langle \nabla u, \partial_t F(t,p) \rangle = \left\langle \nabla u, \frac{\nu(F(t,p))}{\mathrm{H}(F(t,p))} \right\rangle = \frac{1}{\mathrm{H}(F(t,p))} \left\langle \nabla u, \frac{\nabla u}{|\nabla u|} \right\rangle,$$

hence,

$$\boxed{\mathrm{H} = |\nabla u|.}$$

Keeping in mind that $\mathrm{H} = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \Delta_1 u$, we derive the level set formulation of IMCF.

**Level set formulation**

Given $\Omega \subseteq M$ closed and bounded, we define $w_1$ as the solution to

$$\begin{cases} \Delta_1 w_1 = |\nabla w_1| & \text{on } M \setminus \Omega, \\ w_1 = 0 & \text{on } \partial\Omega, \\ w_1 \to +\infty & \text{as } \mathrm{d}(x, \Omega) \to +\infty, \end{cases}$$

Given $\Omega \subseteq M$ closed and bounded, we define $u_2$ as the solution to

$$\begin{cases} \Delta u_2 = 0 & \text{on } M \setminus \Omega, \\ u_2 = 1 & \text{on } \partial\Omega, \\ u_2 \to 0 & \text{as } \mathrm{d}(x,\Omega) \to +\infty, \end{cases}$$

If a solution exists and $\Omega$ is a regular domain, then $u_2 \in C^\infty(M \setminus \mathring{\Omega})$ is unique and it is smooth till the boundary of $\Omega$. By Sard theorem, the set of critical values of $u_2$ has zero Lebesgue measure.

Notice that $w_2 = -\log(u_2)$ solves

$$\begin{cases} \Delta w_2 = |\nabla w_2|^2 & \text{on } M \setminus \Omega, \\ w_2 = 0 & \text{on } \partial\Omega, \\ w_2 \to +\infty & \text{as } \mathrm{d}(x,\Omega) \to +\infty, \end{cases}$$

Suitable variants of the Dirichlet Energy

$$\mathcal{G}_2 = \int_\Sigma |\nabla u|^2 \mathrm{d}\sigma$$

are monotone on the level set of harmonic functions.

Non-exhaustive list of applications:

- Uniqueness of smooth tangent cones at infinity in Ricci-flat manifolds *[Colding '12 · Acta Math.]*,*[Colding, Minicozzi '14 · Invent. Math.]*
- Riemannian positive mass theorem *[Agostiniani, Mazzieri, Oronzio '24 · CMP]*, *[Cederbaum, León Quirós, Meco '25]*
- Willmore inequality *[Agostiniani, Fogagnolo, Mazzieri '20 · Invent. Math.]* *[Cederbaum, Miehe '25 · Trans. Am. Math. Soc.]*
- Characterization of Ricci pinched $3$-manifolds *[Benatti, Mantegazza, Oronzio, **P** —, '25 · J. Geom. An.]*

# 2 | Formal relations

**IMCF**

$$\begin{cases} \partial_t F(p,t) = \frac{\nu(F(t,p))}{\mathrm{H}(F(t,p))} \\ F(0,p) = F_0(p) \end{cases}$$

**Harmonic functions**

$$\begin{cases} \Delta u_2 = 0 & \text{on } M \setminus \Omega, \\ u_2 = 1 & \text{on } \partial\Omega, \\ u_2 \to 0 & \text{as } \mathrm{d}(x,\Omega) \to +\infty, \end{cases}$$

**Reformulation**

Level set formulation

$$\begin{cases} \Delta_1 w_1 = |\nabla w_1| & \text{on } M \setminus \Omega, \\ w_1 = 0 & \text{on } \partial\Omega, \\ w_1 \to +\infty & \text{as } \mathrm{d}(x,\Omega) \to +\infty, \end{cases}$$

Change of variable

$$\begin{cases} \Delta w_2 = |\nabla w_2|^2 & \text{on } M \setminus \Omega, \\ w_2 = 0 & \text{on } \partial\Omega, \\ w_2 \to +\infty & \text{as } \mathrm{d}(x,\Omega) \to +\infty, \end{cases}$$

**Monotone quantities**

Willmore energy $\mathscr{F}_1(t) = \int_{\partial\Omega_t} \mathrm{H}^2 \, \mathrm{d}\sigma$

Hawking mass $\mathfrak{m}_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma \mathrm{H}^2 \, \mathrm{d}\sigma\right)$

Dirichlet energy $\mathscr{G}_2(t) = \int_{\Sigma_t} |\nabla w_2|^2 \, \mathrm{d}\sigma$

## Definition - p-IMCF

Let $p \in [1,2]$. Given $\Omega \subseteq M$ closed and bounded, we define $w_p$ as the solution to

$$\begin{cases} \Delta_p w_p = |\nabla w_p|^p & \text{on } M \setminus \Omega, \\ w_p = 0 & \text{on } \partial\Omega, \\ w_p \to +\infty & \text{as } \mathrm{d}(x,\Omega) \to +\infty, \end{cases} \tag{p-IMCF}$$

where $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2} \nabla f)$.

We denote $\Omega_t^{(p)} = \{w_p \le t\}$.

- $p = 2$: linear potential theory, $u_2 = \exp(-w_2)$ is harmonic.
- $p \in (1,2)$: nonlinear potential theory (NPT), $u_p = \exp(-w_p/(p-1))$ is $p$-harmonic.
- $p = 1$: weak inverse mean curvature flow *[Huisken, Ilmanen '01 · JDG]*.

## Proposition - *[Moser '07 · JEMS], [Kotschwar, Ni '09 · Ann. Sci. Éc. Norm. Supér.]*

If $w_p$ exists for every $p \in [1,2]$, then $w_p \to w_1$ locally uniformly as $p \to 1^+$.

Aim

We want to approximate monotonic quantities along the IMCF with monotonic quantities in NPT.

**Step 1:** Approximation of the Willmore energy

### Construction of $\mathscr{F}_p$ proxy for $\mathscr{F}_1 = \int_{\Sigma_t} \mathrm{H}^2 \mathrm{d}\sigma$.

Formally,

- $\mathscr{F}_p$ should coincide with $\mathscr{F}_1$ in the limit $p \to 1^+$.
- $\mathscr{F}'_p$ should coincide with $\mathscr{F}'_1$ in the limit $p \to 1^+$.
- $\mathscr{F}_p$ should be monotone along $\Delta_p w_p = |\nabla w_p|^p$.

APPROXIMATION

Consider as model case $M=\mathbb{R}^3$ and $\Omega$ with radial symmetry. Call $w_p$ and $w_1$ the solutions to $\Delta_p w_p = |\nabla w_p|^p$ and $\Delta_1 w_1 = |\nabla w_1|$, respectively.

Then, $w_p=(3-p)\log|x|$, $w_1 = 2\log|x|$ and the relation $\frac{2}{3-p} w_p = w_1$ is satisfied. Hence

$$\frac{2}{3-p}|\nabla w_p| = |\nabla w_1| = \mathrm{H}.$$

The term $\left(\mathrm{H}-\frac{2}{3-p}|\nabla w_p|\right)^2$ is a sort of deficit from the model case of IMCF in $\mathbb{R}^3$.

Possible candidates are

$$\mathscr{F}_p(t) = \int_{\partial\Omega_t} \mathrm{H}^2 - \left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\sigma \quad \text{and} \quad \mathscr{G}_p(t) = \int_{\partial\Omega_t} \frac{4|\nabla w_p|^2}{(3-p)^2}.$$

PROXY FOR $\mathscr{F}_1$

From direct computation we have

$$\mathcal{F}_1'(t) = -2\int_{\partial\Omega_t} \frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2} + |\mathring{\mathrm{h}}|^2 + \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\sigma,$$

$$\mathcal{F}_p'(t) = -\frac{4}{3-p}\int_{\partial\Omega_t} \frac{|\nabla^\top|\nabla w_p||^2}{|\nabla w_p|^2} + |\mathring{\mathrm{h}}|^2 + \mathrm{Ric}(\nu,\nu) + \frac{3-p}{2(p-1)}\left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\sigma$$

$$\mathcal{G}_p'(t) = \frac{4}{p-1}\int_{\partial\Omega_t} 2\frac{|\nabla w_p|^2}{(3-p)^2} - \frac{|\nabla w_p|\mathrm{H}}{3-p}\,\mathrm{d}\sigma = \frac{1}{p-1}(\mathcal{G}_p(t) - \mathcal{F}_p(t))$$

If $\mathrm{Ric} \geq 0$, then $\mathcal{F}_p'(t) \leq 0$. Moreover,

$$\left(\mathrm{e}^{-\frac{1}{p-1}t}\mathcal{G}_p'(t)\right)' = \mathrm{e}^{-\frac{1}{p-1}t}\left(\mathcal{G}_p''(t) - \frac{1}{p-1}\mathcal{G}_p'(t)\right) = -\frac{\mathrm{e}^{-\frac{1}{p-1}t}}{p-1}\mathcal{F}_p'(t) \geq 0.$$

If, in addiction, $\mathrm{AVR}(g) > 0$, this implies $\mathcal{G}_p'(t) \leq 0$, thus

$$0 \leq \mathcal{G}_p(t) \leq \mathcal{F}_p(t) \leq \mathcal{F}_1(t)$$

$\mathcal{F}_p$ is monotone non-increasing.

**Step 2:** "p-version" of the Hawking mass $\mathfrak{m}_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \mathrm{H}^2 \, \mathrm{d}\sigma\right)$. We need to

- approximate the area term, maintaining its scaling;
- maintain the exponetial growth along the flow.

**Definition - Normalized $p$-capacity**

Let $\Omega$ be a closed, bounded set in $M$. The normalized $p$-capacity of $\Omega$ is defined as

$$\mathfrak{c}_p(\partial\Omega) := \inf\left\{ \frac{1}{4\pi}\left(\frac{p-1}{3-p}\right)^{p-1} \int_{M\setminus\Omega} |\nabla\phi|^p \, \mathrm{d}\mu \,:\, \phi \in \mathscr{C}_c^\infty(M), \phi \geq 1 \text{ on } \Omega \right\}$$

It holds $\mathfrak{c}_p(\partial\Omega_t) = \mathrm{e}^t \, \mathfrak{c}_p(\partial\Omega)$ and $\mathfrak{c}_p(\partial\Omega) = \int_{\partial\Omega} \left(\frac{|\nabla w_p|}{3-p}\right)^{p-1} \mathrm{d}\sigma$.

As $p \to 1^+$ we have that $\mathfrak{c}_p(\Sigma)$ tends to $|\Sigma^*|$, where $\Sigma^*$ is the outward minimizing hull of $\Sigma$ *[Fogagnolo, Mazzieri '22 · JFA]*.

Natural choice:

$$\mathfrak{m}_H^{(p)}(\Sigma) := \frac{\mathfrak{c}_p(\Sigma)^{\frac{1}{3-p}}}{2}\left(1 - \frac{1}{16\pi}\int_\Sigma \mathrm{H}^2 - \left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\sigma\right)$$

P-HAWKING MASS

Finally, we would like to "summarize" **setting 1** and **setting 2** in the same functional, that then can be "specialized" both to Willmore-type energies and Hawking-type masses.

$$\mathscr{F}_p(\partial\Omega_t) = \mathfrak{c}_p(\partial\Omega_t)^{\frac{\alpha}{n-p}-1} \int_{\partial\Omega_t} |\nabla w_p|^{\alpha+p-2} \left[ \mathrm{H} - \left( \frac{n-1}{n-p} - \frac{1}{\alpha} \right) |\nabla w_p| \right] \mathrm{d}\sigma$$
$$+ \int_0^t \mathfrak{c}_p(\partial\Omega_s)^{\frac{\alpha}{n-p}-1} \int_{\partial\Omega_s} |\nabla w_p|^{\alpha+p-3} \mathrm{Ric}(\nu,\nu) \mathrm{d}\sigma \mathrm{d}s.$$

With this correction term, we have a monotone functional in $(M,g)$ noncompact, complete, Riemannian manifold of dimension $n \geq 3$, without restrictions nor on $\mathrm{Ric}$ neither on $\mathrm{R}$ of $M$.

**p-IMCF** | **p-IMCF**

$$\begin{cases}\Delta_p w_p = |\nabla w_p|^p & \text{on } M \setminus \Omega,\\ w_p = 0 & \text{on } \partial\Omega,\\ w_p \to +\infty & \text{as } \mathrm{d}(x,\Omega)\to+\infty,\end{cases}$$

$$\begin{cases}\Delta_1 w_1 = |\nabla w_1| & \text{on } M \setminus \Omega,\\ w_1 = 0 & \text{on } \partial\Omega,\\ w_1 \to +\infty & \text{as } \mathrm{d}(x,\Omega)\to+\infty,\end{cases}$$

**As $p \to 1^+$, solutions converge to solutions and level sets to level sets.**

## EXPECTATIONS

$$\mathscr{F}_p(t) = \int_{\partial\Omega_t} \mathrm{H}^2 - \left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\sigma$$

$$\mathscr{F}_p'(t) = -\frac{4}{3-p}\int_{\partial\Omega_t} \frac{|\nabla^\top|\nabla w_p||^2}{|\nabla w_p|^2} + |\mathring{\mathrm{h}}|^2 + \mathrm{Ric}(\nu,\nu) + \frac{3-p}{2(p-1)}\left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\sigma$$

$$\mathscr{F}_1(t) = \int_{\partial\Omega_t} \mathrm{H}^2\, \mathrm{d}\sigma$$

$$\mathscr{F}_1'(t) = -2\int_{\partial\Omega_t} \frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2} + |\mathring{\mathrm{h}}|^2 + \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\sigma$$

**$\mathscr{F}_p(t)$ approximate $\mathscr{F}_1(t)$, also at the level of their derivatives.**

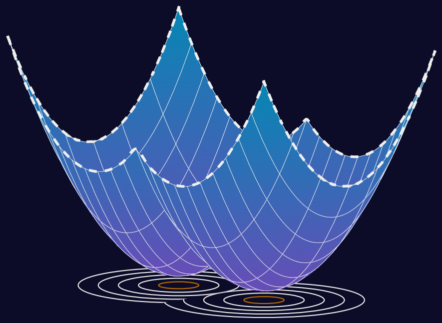
# 3 | Limitations of the smooth setting

**Theorem** – *[Gerhardt '90 · JDG],[Urbas '90 · Math.Z.]*

Any compact, star-shaped initial surface remains star-shaped and smooth along the flow and becomes an expanding round sphere as $t \to +\infty$.

The classical IMCF could develop singularities.

Example:



The level set formulation $\Delta_1 w_1 = |\nabla w_1|$

- is a degenerate elliptic PDE,
- is not the Euler-Lagrange equation of a functional.

Existence and regularity theory for weak IMCF is delicate *[Huisken, Ilmanen '01 · JDG]*.

| p-IMCF | p-IMCF |
|---|---|
| **Regularity** | |
| $w_p \in \mathscr{C}^{1,\beta}$ and smooth on $\{\nabla w_p \neq 0\}$<br>$\lvert\nabla w_p\rvert \in W^{1,2}$ | $w_1$ is Lipschitz |
| **Regularity of level sets** | |
| $\lvert\nabla w_p\rvert \neq 0$ almost everywhere on $\partial\Omega_t$ for a.e. $t$<br>$\partial\Omega_t$ is smooth away from the critical set. | $\Omega_t$ is strictly outward minimizing<br>$\partial\Omega_t$ admits a weak mean curvature $\mathrm{H} > 0$<br>$\partial\Omega_t$ is $\mathscr{C}^{1,\beta}$ out of a set of dimension $n-8$. |

**Goal:** Interpret the level sets as (weak) surfaces, regular enough to get a notion of mean curvature, second fundamental form, and Gauss curvature.

# 4 | GMT meets PDE

$$\mathcal{F}_p'(t) = -\frac{4}{3-p}\int_{\partial\Omega_t^{(p)}} \frac{|\nabla^\top|\nabla w_p||^2}{|\nabla w_p|^2} + |\mathring{\mathrm{h}}|^2 + \mathrm{Ric}(\nu,\nu) + \frac{3-p}{2(p-1)}\left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\sigma$$

$$\mathcal{F}_1'(t) = -2\int_{\partial\Omega_t^{(1)}} \frac{|\nabla^\top|\nabla w_p||^2}{|\nabla w_p|^2} + |\mathring{\mathrm{h}}|^2 + \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\sigma$$

- **Goal 1:** give meaning to all the terms in the formal derivative.
- **Goal 2:** (strong enough) convergence result.

**Theorem** - *[Benatti, **P**—, Pozzetta '24]*

For every $p \in [1,2]$, almost every $\partial\Omega_t^{(p)}$ is a curvature varifold.

Moreover,

- $\partial\Omega_t^{(p)}$ converges (up to subsequence) to $\partial\Omega_t^{(1)}$ in the sense of curvature varifolds for a.e. $t > 0$.
- $w_p \to w_1$ in $W^{1,q}_{\mathrm{loc}}$ for $q < +\infty$.

**Step 1.** almost every $\partial\Omega_t^{(p)}$ is a curvature varifold.

$\varepsilon$-regularization and *a priori* bounds. This gives meaning to all terms in

$$\mathcal{F}_p'(t) = -\frac{4}{(3-p)}\int_{\partial\Omega_t^{(p)}} \frac{\left|\nabla^\top|\nabla w_p|\right|^2}{|\nabla w_p|^2} + |\mathring{\mathrm{h}}|^2 + \mathrm{Ric}(\nu,\nu) + \frac{3-p}{2(p-1)}\left(\mathrm{H} - \frac{2}{3-p}|\nabla w_p|\right)^2 \mathrm{d}\sigma.$$

**Step 2.** $\partial\Omega_t^{(p)}$ converges to $\partial\Omega_t^{(1)}$ in the sense of curvature varifolds for a.e. $t>0$.

1. $|\partial\Omega_t^{(p)}| \xrightarrow{p\to1^+} |\partial\Omega_t^{(1)}|$ for almost every $t$.
2. $\displaystyle\int_{\partial\Omega_t^{(p)}} (|\nabla w_p| - \mathrm{H})^2 \mathrm{d}\sigma \xrightarrow{p\to1^+} 0.$
3. $\displaystyle\int_{\partial\Omega_t^{(p)}} |\mathrm{h}|^2 \mathrm{d}\sigma$ is bounded.

1. + 2. $\Rightarrow$ convergence in the sense of varifolds.

2. + bound on $\int |\mathring{\mathrm{h}}|^2 \Rightarrow$ 3.

1. + 2. + 3. $\Rightarrow$ convergence as curvature varifolds.

**Step 3.** Gradient convergence.

Combine:

- $\nabla w_p \xrightharpoonup{p\to1^+} \nabla w_1$ weakly in $L^q$
- $\|w_p\|_{L^1_{\mathrm{loc}}} \xrightarrow{p\to1^+} \|w_1\|_{L^1_{\mathrm{loc}}}$
- $\displaystyle\lim_{p\to1^+}\int_0^T\int_{\partial\Omega_t^{(p)}} |\nu^{(p)} - \nu^{(1)}|^2 \mathrm{d}\sigma\mathrm{d}t = 0$

For $p \in (1,2)$, we do not have enough regularity to define an Euler characteristic that encompass the topological properties of the level sets.

A weak induced scalar curvature can be defined through an integral version of the Gauss equation:

$$\int_\Sigma \mathrm{R}^\top \mathrm{d}\sigma = \int_\Sigma \mathrm{R} - 2\mathrm{Ric}(\nu,\nu) + \mathrm{H}^2 - |\mathrm{h}|^2 \mathrm{d}\sigma$$

**Theorem** - *[Benatti,* ***P*** *—, Pozzetta '24]*

Let $(M,g)$ be a complete, noncompact 3-dimensional Riemannian manifold. Let $p \in (1,2)$ and $w_p$ be the solution to $\Delta_p w_p = |\nabla w_p|^p$. Then, for almost every $t \in [0,+\infty)$ it holds

$$\int_{\partial\Omega_t^p} \mathrm{R}^\top \mathrm{d}\sigma \in 8\pi\mathbb{Z}.$$