From linear potential theory to the inverse mean curvature flow

Applications to Riemannian Penrose-type inequalities

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Geometric Measure Theory on Metric Spaces with Applications to Physics and Geometry

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INITIAL DATA SETS

- A **time-symmetric initial data set** is a Riemannian 3-manifold $(M, g, K=0)$ satisfying the Einstein constraint equations

$$8\pi\mu = \frac{1}{2}\left(\mathrm{R} + \mathrm{tr}K^2 - |K|^2\right) = \frac{1}{2}\mathrm{R}, \qquad \text{(energy density)}$$

$$8\pi J = \mathrm{div}(K - \mathrm{tr}K\, g) = 0. \qquad \text{(momentum density)}$$

- The **dominant energy condition** prescribes $\mathrm{R} \geq 0$.
- An outermost **apparent horizon** Σ is an outermost minimal surface.

Definition - Asymptotically flat manifold

(M, g) is $\mathscr{C}^k_\tau$-asymptotically flat provided $M \setminus K \cong \mathbb{R}^3 \setminus B_R$ and $|g - \delta| = O_k(|x|^{-\tau})$.

(M, g) arises as a spacelike time slice in an isolated gravitating system governed by Einstein's equations.

$$\sqrt{\frac{|\Sigma|}{16\pi}} \leq \mathfrak{m}$$

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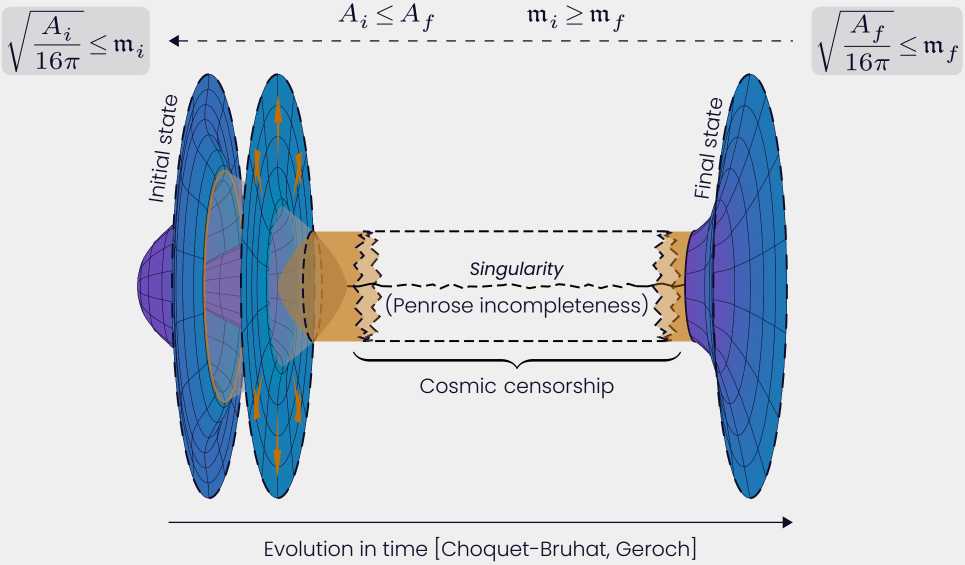
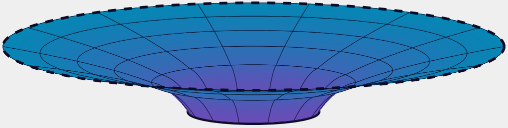
PENROSE'S ARGUMENT



Spatial Schwarzschild manifold

Given $\mathfrak{m} \geq 0$, it is

$$\left(\mathbb{R}^3 \smallsetminus B_{2\mathfrak{m}}, \left(1 + \frac{\mathfrak{m}}{2|x|}\right)^4 \delta\right).$$



- It is scalar flat $\mathrm{R} = 0$.
- ∂M is outermost minimal.
- It is $\mathscr{C}^\infty_1$-asymptotically flat.

The quantity $\mathfrak{m}$ represents the mass of the black hole and <u>satisfies the equality</u> in the Penrose inequality

$$\mathfrak{m} = \sqrt{\frac{|\partial M|}{16\pi}}.$$

Definition - ADM mass *[Arnowitt, Deser, Misner '61 · Phys. Rev.]*

$$\mathfrak{m}_{\mathrm{ADM}} := \lim_{R\to+\infty} \frac{1}{16\pi} \int_{\partial B_R} (\partial_j g_{kj} - \partial_k g_{jj}) \frac{x^k}{|x|} \mathrm{d}\sigma_\delta$$

The naive idea would be to integrate the mass density $\mu = \mathrm{R}/(16\pi)$

$\Longrightarrow$ spatial Schwarzschild would have 0 mass, and

$\Longrightarrow$ there would be no superposition principle.

$$\frac{1}{16\pi}\int_M \mathrm{R}\,\mathrm{dVol} \approx \frac{1}{16\pi}\int_M \nabla \mathrm{R}_{|_\delta}(g-\delta)\,\mathrm{d}x = \lim_{R\to+\infty}\frac{1}{16\pi}\int_{B_R} \partial_k(\partial_j g_{kj} - \partial_k g_{jj})\,\mathrm{d}x.$$

Theorem - *[Bartnik '86 · CPAM], [Chruściel '86 · SPRINGER]*

$\mathfrak{m}_{\mathrm{ADM}}$ is a geometric invariant if (M,g) is $\mathscr{C}^1_\tau$-asymptotically flat, $\tau > 1/2$.

- The general case is still a conjecture.

Theorem - Riemannian Penrose inequality

Let (M, g) be a $\mathscr{C}^1_1$-asymptotically flat 3-Riemannian manifold with $\mathrm{R} \geq 0$ and $\mathrm{Ric} \geq -\mathrm{C}/|x|^2$ and a minimal connected outermost boundary ∂M. Then,

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \mathfrak{m}_{\mathrm{ADM}}.$$

Moreover, the equality holds <u>if and only if</u> (M, g) is isometric to the Schwarzschild of mass $\mathfrak{m}_{\mathrm{ADM}}$.

- *[Bray '01 · JDG]* for disconnected horizons (up to dimension 8 *[Bray, Lee '09 · DUKE]*).
- *[Huisken, Ilmanen '01 · JDG]* for connected horizons using inverse mean curvature flow (IMCF).
- *[Agostiniani, Mantegazza, Mazzieri, Oronzio '22]* for connected horizons using nonlinear potential theory.

Prove the Riemannian Penrose inequality for $\mathscr{C}^1_{\tau > 1/2}$-asymptotically flat manifolds.

1 Bridging two paths towards the Riemannian Penrose inequality

- Consider the **Hawking mass**

$$\mathfrak{m}_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}}\,\overbrace{\left(1 - \frac{1}{16\pi}\int_\Sigma \mathrm{H}^2\,\mathrm{d}\sigma\right)}^{\text{Willmore deficit in } \mathbb{R}^3}.$$

  - ∂M is minimal $\quad\Longrightarrow\quad \mathfrak{m}_H(\partial M) = \sqrt{\frac{|\partial M|}{16\pi}}$

- **Evolve ∂M using the weak IMCF w_1** and define $\Sigma_t^{(1)} := \partial\{w_1 \le t\}$.

  - M is 3-dimensional and $\mathrm{R} \ge 0$

$$\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H(\Sigma_t^{(1)}) \ge \frac{1}{16\pi}\sqrt{\frac{|\Sigma_t^{(1)}|}{16\pi}}\left(\overbrace{8\pi - \int_{\Sigma_t^{(1)}} \mathrm{R}^\top\,\mathrm{d}\sigma}^{\text{Gauss-Bonnet}} + \int_{\Sigma_t^{(1)}} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + \frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2}\,\mathrm{d}\sigma\right) \ge 0.$$

$\Longrightarrow \quad t \mapsto \mathfrak{m}_H(\Sigma_t^{(1)})$ is monotone <u>non decreasing</u>.

  - g is asymptotically flat

$\Longrightarrow \quad \Sigma_t^{(1)}$ asymptotically approaches large coordinate spheres as $t \to +\infty$,

$$\Longrightarrow \quad \lim_{t\to+\infty} \mathfrak{m}_H(\Sigma_t^{(1)}) \le \mathfrak{m}_{\mathrm{ADM}}.$$

We introduce the p-**Hawking mass**, which is

$$\mathfrak{m}_H^{(p)}(\Sigma) := \frac{\mathfrak{c}_p(\Sigma)^{\frac{1}{3-p}}}{8\pi}\left(\overbrace{4\pi - \int_\Sigma \frac{\mathrm{H}^2}{4}\,d\sigma}^{\text{Willmore deficit in } \mathbb{R}^3} + \int_\Sigma \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 d\sigma\right),$$

where $\mathfrak{c}_p(\Sigma)$ is the p-capacity of Σ and w_p is the p-IMCF issuing from Σ.

- The 1-Hawking mass is the Hawking mass for strictly outward minimizing Σ:

$$\mathfrak{m}_H^{(1)}(\Sigma) = \frac{1}{4\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\left(4\pi - \int_\Sigma \frac{\mathrm{H}^2}{4}\,d\sigma + \cancel{\int_\Sigma \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_1|}{2}\right)^2 d\sigma}\right) = \mathfrak{m}_H(\Sigma)$$

- The Hawking mass is the smallest:

$$\mathfrak{m}_H(\Sigma) \leq \frac{1}{4\pi}\sqrt{\frac{|\Sigma|}{4\pi}}\left(4\pi - \int_\Sigma \frac{\mathrm{H}^2}{4}\,d\sigma + \int_\Sigma \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 d\sigma\right) = \sqrt{\frac{|\Sigma|}{4\pi}}\,\mathfrak{c}_p(\Sigma)^{-\frac{3}{3-p}}\,\mathfrak{m}_H^{(p)}(\Sigma)$$

$$\mathfrak{m}_H^{(q)}(\Sigma) \not\leq \mathfrak{c}_p(\Sigma)^{\frac{1}{3-q}}\,\mathfrak{c}_p(\Sigma)^{-\frac{1}{3-p}}\,\mathfrak{m}_H^{(p)}(\Sigma) \qquad \text{for } q \neq 1$$

MONOTONICITY FORMULAS

Theorem - *[B —, Pluda, Pozzetta '24]*

Let w_p be the proper p-IMCF issuing from Σ, $p \in [1,2]$, and $\Sigma_t^{(p)} = \partial\{w_p \le t\}$. Then, $t \mapsto \mathfrak{m}_H^{(p)}(\Sigma_t^{(p)})$ is monotone nondecreasing and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \ge \frac{\mathfrak{c}_p (\Sigma_t^{(p)})^{\frac{1}{3-p}}}{16\pi(3-p)}\left(8\pi - \int_{\Sigma_t^{(p)}} \mathrm{R}^\top \mathrm{d}\sigma + \int_{\Sigma_t^{(p)}} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + \frac{\left|\nabla^\top |\nabla w_p|\right|^2}{|\nabla w_p|^2} + \frac{5-p}{p-1}\left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 \mathrm{d}\sigma\right).$$

Moreover,

- $\underline{\lim}_{p \to 1^+} \mathfrak{m}_H^{(p)}(\Sigma) \ge \mathfrak{m}_H(\Sigma)$,
- $\mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \to \mathfrak{m}_H(\Sigma_t^{(1)})$ in L^1_{loc} as $p \to 1^+$,
- and the above inequality passes to the limit.

- For $p > 1$, monotonicity was proven in *[Agostiniani, Mantegazza, Mazzieri, Oronzio '22]*.
- For $p = 1$, the formula was proven in *[Huisken, Ilmanen '01 · JDG]*.

$$\lim_{t\to+\infty} \frac{\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}}{8\pi}\left(4\pi - \int_{\Sigma_t^{(p)}} \frac{\mathrm{H}^2}{4}\,\mathrm{d}\sigma + \int_{\Sigma_t^{(p)}} \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 \mathrm{d}\sigma\right)$$

- Asymptotic behaviors of w_p and g come into play.
- The better the asymptotics for w_p, the fewer the constraints on g.

Characterize $\varepsilon_p(x) := w_p(x) - \overbrace{(3-p)\log|x| - (3-p)\log \mathfrak{c}_p(\Sigma)}^{\text{model solution on } \mathbb{R}^3}$

$p=1$

$\mathfrak{m}_H$ depends only on $|\nabla w_1| = \mathrm{H}$.

$\mathscr{C}^1_\tau$-asymptotic flatness and $\mathrm{Ric} \geq -\mathrm{C}|x|^{-2} \implies \varepsilon_1(x) = o(1)$ and $\Sigma_t^{(1)} \sim \mathbb{S}^2(\mathrm{e}^{\frac{t}{2}})$ in $\mathscr{C}^1$.

$\mathscr{C}^1_1$-asymptotic flatness and $\mathrm{Ric} \geq -\mathrm{C}|x|^{-2} \implies \lim_{t\to+\infty} \mathfrak{m}_H(\Sigma_t^{(1)}) \leq \mathfrak{m}_{\mathrm{ADM}}$.

$p>1$

$\mathfrak{m}_H^{(p)}$ depends aslo on $\nabla|\nabla w_p|$.

$\mathscr{C}^1_\tau$-asymptotic flatness $\implies \varepsilon_p(x) = o_1(1)$.

$p=2$

$\mathscr{C}^1_\tau$-asymptotic flatness $\implies \varepsilon_2(x) = o_1(|x|^{-\tau})$ and $\|\mathrm{Hess}\,\varepsilon_2\|_{L^2(\Sigma_t^{(2)})} = O(\mathrm{e}^{-2\tau t})$.

$\mathscr{C}^1_{\tau>1/2}$-asymptotic flatness $\implies \lim_{t\to+\infty} \mathfrak{m}_H^{(2)}(\Sigma_t^{(2)}) \leq \mathfrak{m}_{\mathrm{ADM}}$.

ASYMPTOTIC BEHAVIOR

Assume (M,g) is $\mathscr{C}^1_{\tau>1/2}$-asymptotically flat

$$\mathfrak{m}_H(\partial M) \leq \lim_{t\to+\infty} \mathfrak{m}_H(\Sigma_t^{(1)}) \leq \varlimsup_{t\to+\infty} \sqrt{\frac{|\Sigma_t^{(1)}|}{4\pi}}\,\mathfrak{c}_2(\Sigma_t^{(1)})^{-1}\,\mathfrak{m}_H^{(2)}(\Sigma_t^{(1)}) \leq \mathrm{C}\,\mathfrak{m}_{\mathrm{ADM}}.$$

$$\varepsilon_p(x) = o(1) \quad\Longrightarrow\quad \mathfrak{c}_q(\Sigma_t^{(p)})^{\frac{1}{3-q}}\,\mathfrak{c}_p(\Sigma_t^{(p)})^{-\frac{1}{3-p}} = (1+o(1)).$$

Assuming $\mathrm{Ric} \geq -\mathrm{C}|x|^{-2} \quad\Longrightarrow\quad \mathfrak{m}_H(\partial M) \leq \mathfrak{m}_{\mathrm{ADM}}$

$\mathscr{C}^1_{\tau>1/2}$-asymptotic flatness $\quad\Longrightarrow\quad \varepsilon_p(x) = o_1(1)$ for $p>1$.

$$\mathfrak{m}_H^{(p)}(\Sigma) \not\leq \mathfrak{c}_p(\Sigma)^{\frac{1}{3-p}}\,\mathfrak{c}_2(\Sigma)^{-1}\,\mathfrak{m}_H^{(2)}(\Sigma)$$ for $p \neq 1$

$$\mathfrak{m}_H^{(p)}(\partial M) \leq \lim_{t\to+\infty} \mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \leq \varlimsup_{t\to+\infty} \mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}\,\mathfrak{c}_2(\Sigma_t^{(p)})^{-1}\,\mathfrak{m}_H^{(2)}(\Sigma_t^{(p)}) \leq \mathfrak{m}_{\mathrm{ADM}},$$

and $\varliminf_{p\to1^+} \mathfrak{m}_H^{(p)}(\partial M) \geq \sqrt{\dfrac{|\partial M|}{16\pi}}$.

MAIN IDEA

$$\lim_{t\to+\infty} \mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \leq \overline{\lim_{t\to+\infty}} \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(4\pi - \int_{\Sigma_t^{(p)}} \frac{\mathrm{H}^2}{4} - \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 \mathrm{d}\sigma\right)}{8\pi \frac{\mathrm{d}}{\mathrm{d}t}\, \mathfrak{c}_p(\Sigma_t^{(p)})^{-\frac{1}{3-p}}}$$ (de l'Hôpital rule)

$$\leq \overline{\lim_{t\to+\infty}} \frac{\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}}{8\pi}\left(\int_{\Sigma_t^{(p)}} \frac{\mathrm{R}^\top}{2} - \frac{\mathrm{H}^2}{4}\,\mathrm{d}\sigma\right)$$ (computing derivatives)

$$\leq \overline{\lim_{t\to+\infty}} \frac{\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}}{8\pi}\left(4\pi - \int_{\Sigma_t^{(p)}} \frac{\mathrm{H}^2}{4}\,\mathrm{d}\sigma\right)$$ (Gauss–Bonnet theorem)

$$= \overline{\lim_{t\to+\infty}}\, \mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}} \sqrt{\frac{4\pi}{|\Sigma_t^{(p)}|}}\, \mathfrak{m}_H(\Sigma_t^{(p)})$$ (recognize $\mathfrak{m}_H$)

$$\leq \overline{\lim_{t\to+\infty}}\, \mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}\, \mathfrak{c}_2(\Sigma_t^{(p)})^{-1} \mathfrak{m}_H^{(2)}(\Sigma_t^{(p)}) \leq \mathfrak{m}_{\mathrm{ADM}}$$ ($\mathfrak{m}_H$ is the smaller)

Theorem - *[B** —, Fogagnolo, Mazzieri '25 · CPAM]***

Let (M, g) be a $\mathscr{C}^1_\tau$-asymptotically flat 3-Riemannian manifold, $\tau > 1/2$, with $\mathrm{R} \geq 0$ and a minimal connected outermost boundary. Then,

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \mathfrak{m}_{\mathrm{ADM}}.$$

Moreover, the equality holds <u>if and only if</u> (M, g) is isometric to the Schwarzschild of mass $\mathfrak{m}_{\mathrm{ADM}}$.

2 Beyond the horizon of ADM mass toward milder asymptotics

[Huisken '09] introduced the notion of **isoperimetric mass**, which is

$$\mathfrak{m}_{\mathrm{iso}} := \sup_{\Omega_k} \varlimsup_{\Omega_k \uparrow M} \mathfrak{m}_{\mathrm{iso}}(\Omega_k),$$

where Ω_k is an exhaustion of M and

$$\mathfrak{m}_{\mathrm{iso}}(\Omega_k) := \frac{2}{|\partial \Omega_k|} \Bigg(\underbrace{|\Omega_k| - \frac{|\partial \Omega_k|^{\frac{3}{2}}}{6\sqrt{\pi}}}_{\text{isoperimetric deficit in } \mathbb{R}^3} \Bigg).$$

- $\mathfrak{m}_{\mathrm{iso}}$ is a well-defined geometric invariant and depends only on the notions of volume and area.
- $\mathfrak{m}_{\mathrm{iso}} \geq 0$ by *[Jauregui, Lee, Unger '24]*, even if $\mathrm{R} \geq 0$ is not satisfied.
- $\mathfrak{m}_{\mathrm{iso}} \geq \mathfrak{m}_{\mathrm{ADM}}$ by *[Fan, Shi, Tam '09 · CAG]*
  $\Longrightarrow$ (RPI) holds under stronger assumptions.
- $\mathfrak{m}_{\mathrm{iso}} \leq \mathfrak{m}_{\mathrm{ADM}}$ by combining *[Jauregui, Lee '19 · CRELLE]* and *[**B** —, Fogagnolo, Mazzieri '25 · CPAM]*
  $\Longrightarrow$ *[Bartnik '86 · CPAM]* and *[Chruściel '86 · SPRINGER]*.

Theorem - [*B* —, Fogagnolo, Mazzieri '25 · CPAM]

Let (M,g) be a $\mathscr{C}^0$-asymptotically flat 3-Riemannian manifold, with $\mathrm{R} \geq 0$ and a minimal connected outermost boundary. Then,

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \mathfrak{m}_{\mathrm{iso}}.$$

Moreover, the equality holds if and only if (M,g) is isometric to the Schwarzschild of mass $\mathfrak{m}_{\mathrm{iso}}$.

Let $\Omega_t^{(1)} = \{w_1 \le t\}$ (so that $\partial\Omega_t^{(1)} = \Sigma_t^{(1)}$).

$$\lim_{t\to+\infty} \mathfrak{m}_{\mathrm{iso}}(\Omega_t^{(1)}) \ge \lim_{t\to+\infty} 2\,\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(|\Omega_t^{(1)}| - \frac{|\Sigma_t^{(1)}|^{\frac{3}{2}}}{6\sqrt{\pi}}\right)}{\frac{\mathrm{d}}{\mathrm{d}t}|\Sigma_t^{(1)}|} \qquad \text{(de l'Hôpital rule)}$$

$$= \lim_{t\to+\infty} \frac{2}{|\Sigma_t^{(1)}|}\left(\int_{\Sigma_t^{(1)}} \frac{1}{\mathrm{H}}\,\mathrm{d}\sigma - \frac{|\Sigma_t^{(1)}|^{\frac{3}{2}}}{4\sqrt{\pi}}\right) \qquad \text{(computing derivatives)}$$

$$\ge \lim_{t\to+\infty} \frac{2}{|\Sigma_t^{(1)}|}\left(|\Sigma_t^{(1)}|^{\frac{3}{2}}\left(\int_{\Sigma_t^{(1)}} \mathrm{H}^2\,\mathrm{d}\sigma\right)^{-\frac{1}{2}} - \frac{|\Sigma_t^{(1)}|^{\frac{3}{2}}}{4\sqrt{\pi}}\right) \qquad \text{(Hölder's inequality)}$$

$$= \lim_{t\to+\infty} 2\left(\frac{|\Sigma_t^{(1)}|}{\int_{\Sigma_t^{(1)}} \mathrm{H}^2\,\mathrm{d}\sigma}\right)^{\frac{1}{2}}\left[1 - \left(\frac{1}{16\pi}\int_{\Sigma_t^{(1)}} \mathrm{H}^2\,\mathrm{d}\sigma\right)^{\frac{1}{2}}\right] \qquad \text{(factorizing)}$$

$$= \lim_{t\to+\infty} \mathfrak{m}_H(\Sigma_t^{(1)}) \ge \mathfrak{m}_H(\partial M) = \sqrt{\frac{|\partial M|}{16\pi}} \qquad \text{(monotonicity)}\ (\mathfrak{m}_{\mathrm{iso}} < +\infty)$$

ISOCAPACITARY MASS

[Jauregui '20 · CAG] introduced (for $p=2$) the **iso-p-capacitary mass** $\mathfrak{m}^{(p)}_{\mathrm{iso}}$, which is

$$\mathfrak{m}^{(p)}_{\mathrm{iso}}(\Omega_k) = \frac{1}{2p\pi\,\mathfrak{c}_p(\partial\Omega_k)^{\frac{2}{3-p}}}\Bigg(\underbrace{|\Omega_k| - \frac{4\pi}{3}\,\mathfrak{c}_p(\partial\Omega_k)^{\frac{3}{3-p}}}_{\text{iso-}p\text{-capacitary deficit in }\mathbb{R}^3}\Bigg).$$

- $\mathfrak{m}^{(p)}_{\mathrm{iso}}$ is a well defined geometric invariant.

What happens when replacing the isoperimetric mass/Hawking mass with the iso-p-capacitary mass/p-Hawking mass in the previous argument?

Theorem - [*B —, Fogagnolo, Mazzieri '23 · SIGMA*]

Let (M,g) be a $\mathscr{C}^0$-asymptotically flat (+ extra assumption) 3-Riemannian manifold, with $\mathrm{R}\geq 0$ and a minimal connected outermost boundary. Then,

$$\mathfrak{c}_p(\partial M)^{\frac{2}{3-p}} \leq \frac{5-p}{2}\mathfrak{m}^{(p)}_{\mathrm{iso}}.$$

- We needed to assume $\int_{\Sigma_t^{(p)}} |\nabla w_p|^2 \mathrm{d}\sigma = o(e^{t/(p-1)})$, because

$$\lim_{t\to+\infty} \mathfrak{m}^{(p)}_{\mathrm{iso}}(\Omega_t^{(p)}) \geq \lim_{t\to+\infty} \frac{\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}}{4\pi(3-p)}\left(4\pi - \int_{\Sigma_t^{(p)}} \frac{|\nabla w_p|^2}{(3-p)^2}\mathrm{d}\sigma\right) \geq \lim_{t\to+\infty} \mathfrak{m}^{(p)}_{H}(\Sigma_t^{(p)}) \geq \frac{1}{2}\mathfrak{c}_p(\partial M)^{\frac{1}{3-p}}.$$

  Does the extra condition hold in general? **(open!)**

- IMCF can jump over minimal horizons. Is it possible to do the same with p-IMCF? **(open!)**
- It is not sharp since $\mathfrak{m}^{(p)}_H$ is constant only on $\mathbb{R}^3$ and not on Schwarzschild spatial manifold. Can $\mathfrak{m}^{(p)}_H$ be replaced with the quantities introduced in *[Xia, Yin, Zhou '24 · Adv. Math.]* to prove the sharp one? **(open!)**

[Bray, Miao '08 · Duke], *[Xiao '16 · Ann. Henri Poincaré]* proved the sharp estimate

$$\mathfrak{c}_p(\partial M)^{\frac{1}{3-p}} \leq k_p\sqrt{\frac{|\partial M|}{16\pi}} \leq k_p \mathfrak{m}_{\mathrm{iso}}.$$

- $\mathfrak{m}_{\mathrm{iso}}^{(2)} = \mathfrak{m}_{\mathrm{ADM}}$ by *[Jauregui '20 · CAG]* under stronger assumptions.
- $\mathfrak{m}_{\mathrm{iso}}^{(p)} = \mathfrak{m}_{\mathrm{ADM}} = \mathfrak{m}_{\mathrm{iso}}$ by *[**B** —, Fogagnolo, Mazzieri '23 · SIGMA]* under sharp assumptions.
- $\mathfrak{m}_{\mathrm{iso}}^{(p)} \to \mathfrak{m}_{\mathrm{iso}}$ in $\mathscr{C}^0$-asymptotically flat manifolds with $\mathrm{R} \geq 0$ by *[**B** —, Fogagnolo, Mazzieri '23 · SIGMA]*.

Theorem - *[B — '2X]*

Let (M,g) be a $\mathscr{C}^0$-asymptotically flat 3-dimensional manifold, $\mathrm{R} \geq 0$. Then, $\mathfrak{m}_{\mathrm{iso}}^{(p)} = \mathfrak{m}_{\mathrm{iso}}$ for every p.