Network flow: resolution of singularities and stability

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Geometric flows are parabolic partial differential equations that aim to deform a geometric objects to simplify them (e.g. to reduce its topological complexity or to make it more symmetric). They have been applied to a variety of topological, analytical and physical problems, giving in some cases very fruitful results. Among other geometric equations the *Mean Curvature Flow* has been extensively studied. This flow can be regarded as the gradient flow of the area functional: a n–dimensional time dependent surface evolves with normal velocity its mean curvature. Nowadays the mean curvature flow of a smooth submanifold is deeply understood. The situation is different when we consider the evolution of generalized, possibly singular, submanifolds. The simplest example of motion by mean curvature of a set which is essentially singular is the *evolution of networks*: 1–dimensional connected sets composed of a finite number of curves that meet at junctions. This is the case of our interest.

Appearing initially in materials science as a model for the evolution of grain boundaries in polycrystals, this flow was later treated by Brakke [1] using varifold methods. A network of curves is a "nearly smooth" object, hence one is tempted to describe its evolution by a direct PDE approach, but doing so requires one to deal with the singular nature of the PDE at the vertices of the network. Interpreting the junctions as boundary points (free to move during the evolution), Bronsard and Reitich [2] proved a *short time existence* result with initial datum a *regular network* composed of three curves meeting at a triple junction forming equal angles. However, this class of networks, is not preserved by the flow: two triple junctions might coalesce while the curvature remains bounded [10], or an enclosed region bounded by a loop of curves in the network might collapse to a point [11].

We would like to know whether this limiting network can evolve past this singular time. Thus the motivation to understand how to start/restart the flow from a network of curves with *"irregular" multiple junctions* goes beyond the basic inherent interest in enlarging the class of admissible initial data. Keeping in mind that stable critical points of the length functional present only triple junctions whose unit tangent vectors at the junctions sum up to zero, we expect to see configurations of this kind for almost all times. We must, in particular, show precisely how a single multi–point gives birth to a cluster of triple junctions.

The short time existence for *irregular* networks [7, 8] reads as follow: let Γ_0 be an initial network of class C^2 and at least one interior vertex is irregular. Then there exists a time $T > 0$ and an evolving family of regular networks Γ_t, $0 < t < T$, with the property that $\Gamma_t \to \Gamma_0$ uniformly. Moreover the set of these possible flowouts is classified by the collection of all *"expanding soliton solutions"* of the flow at each interior vertex.

This results was first proved in [7], but our approach in [8] draws out some important features and precise information not accessible by the previous method. The inspiration for the constructions, that we briefly describe now, comes from the methods of geometric microlocal analysis.

Let $k \geq 3$ and suppose that p is a k–valent irregular vertex of our initial network Γ_0 where the curves $\gamma^{(1)}, \ldots, \gamma^{(k)}$ meet. Denoting by τ_j the unit tangent vector to γ_j at p, the incoming edges at p determine a fan of rays $\ell_j = \mathbb{R}^+\tau_j$ in $\mathbb{R}^2$ emanating from p. There exists at least one (typically many) expanding soliton solution for the network flow which has this fan as its initial condition [15]. Choose one of these solutions, and denote it by S_p. Having made this choice for every irregular interior vertex, we prove that there exists a unique solution Γ_t of the network flow, defined on some interval $0 < t < T$, whose combinatorics are the same as if we were to replace a small ball around each irregular vertex p with the corresponding choice of soliton S_p. This solution converges to Γ_0 as $t \to 0$.

One striking feature is that if there are k–valent irregular vertex, then there exists more than one solution to the network flow emanating from the given initial configuration.

Once established a solid short time existence results, the *long time behavior* of the evolution deserves to be investigated. Suppose that Γ_t is a maximal solution to the network flow in $[0, T)$. Then if T is finite, either the inferior limit of the length of at least one curve of the network $\mathcal{N}(t)$ is zero or the superior limit of the L^2–norm of the curvature of the network is $+\infty$ [12, 6]. There are explicit examples of all these behaviors [11, 16].
Now when $T = +\infty$ we want to understand whether Γ_t converges to a critical point of the length functional. Instead when the flow develops a singularity at a finite time, we aim at a deeper understanding of the singular limit of Γ_t as $t \to T^-$, by means of the analysis of *tangent flows* (sequences of space-time rescalings of the flow that exists for all times). One of the main difficulty is to understand whether the limits as $t \to \infty$ of such rescalings is unique, i.e., whether the tangent flow do not depend on the choice of the rescaling sequence. Hence, both to understand the singularities of the flow and its asymptotic behavior, the key points is the existence of a full limit critical point as t tends to $+\infty$ of certain gradient flows. This issue can be address by means of the so-called *Łojasiewicz–Simon* gradient inequalities introduced in the seminal works [9, 18]. Roughly speaking, an energy functional E satisfies a Łojasiewicz–Simon inequality in the neighborhood U of a critical point if a concave power of the difference in energy between the critical point and a point in U can be bounded from above by a norm of the gradient of E. It turns out that if the given energy E satisfies Łojasiewicz–Simon inequalities in neighborhoods of its critical points, then the gradient flow for E existing for all times converges to a critical point at $t \to +\infty$ (see [3, 13, 14]).
The study of the network flow imposes new difficulties on the application of the aforementioned results carried out for flows of smooth submanifolds, because of the natural singularities of our evolving objects. In [16] we overcome these technical issues and we prove a Łojasiewicz–Simon inequality for the length functional of networks suitably close to minimal ones. This allows us to prove not only the smooth convergence of the network flow whenever it does not develop singularities, but also the stability of minimal networks: for initial data suitably close to minimal networks, the evolution exists for all times without singularities and it smoothly converges to a critical point.
The same method has been applied in [17] to prove the uniqueness of compact blowups of network flow. We plan to investigate the (much more difficult) case of possibly noncompact blowups, also exploiting the recent fundamental achievement on the same problem for mean curvature flow of hypersurfaces [5, 4].

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