Willmore flow of planar networks

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Abstract

We consider networks of curves in the plane moving according to the L^2–gradient flow of the elastic energy. We prove short time existence in the case of networks composed of three curves that are required to meet at one or two triple junctions. As a variation of the result we additionally impose that they form angles of 120 degrees at the triple junction(s).

A planar network $\mathcal{N}$ is a connected set composed of a finite number of sufficiently smooth curves $\mathcal{N}^i$ that meet at junctions. We consider two types of networks of three curves, namely Theta Networks and Triods. A Theta Network consists of three curves that intersect each other at their endpoints in two triple junctions. Each of the three regular curves of a Triod has one endpoint fixed in the plane while the other three endpoints meet at one triple junction. Denoting by s^i and k^i the arc length parameter and curvature of the curve $\mathcal{N}^i$, the elastic energy with global length penalisation is given by

$$E_\mu(\mathcal{N}) := \sum_{i=1}^{3} \int_{\mathcal{N}^i} \left(\left(k^i\right)^2 + \mu \right) \mathrm{d}s^i \,, \qquad \mu \geq 0 \,.$$

We consider the L^2-gradient flow of the energy E_μ: we let evolve an initial network $\mathcal{N}_0$ (Triod or Theta) with a normal velocity that induces the steepest descent of the energy with respect to the L^2–inner product under the constraint that the topology of the initial object is maintained during the flow. The curves of a Theta Network stay attached at two triple junctions which are allowed to move during the flow. In the case of a Triod the curves remain connected at one possibly moving triple junction while the other endpoints are fixed in the plane. Situations in which the angles at the 3–points are not prescribed are called C^0–flow in contrast to the C^1–flow where the curves are required to meet with angles of 120 degrees at the triple junction(s) during the evolution.
The elastic flow for networks was first proposed by Barrett, Garcke and Nürnberg: in [1] it is shown that if a network moves according to the L^2–gradient flow of E_μ, then the normal velocity of each of its curves is given by

$$-2k_{ss} - k^3 + \mu k \,.$$

Depending on whether one imposes an angle condition at the junction(s) (C^1–flow) or not (C^0–flow), different conditions result at the 3–points. In the case of a Triod different scenarios are possible at the fixed endpoints. Either the curvature is zero at the fixed endpoints

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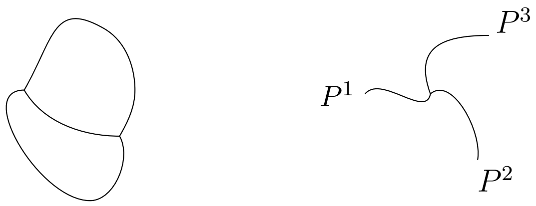

Figure 1: A Theta Network with angle condition and a Triod.

(*Navier*) or we prescribe both the position of the endpoints in the plane and the direction of the tangents of the curves at the fixed endpoints (*clamped*).
To answer the question whether these boundary conditions lead to a well posed evolution problem, we decided to look for classical solutions. Each evolving curve $\mathcal{N}^i$ of the network $\mathcal{N}$ is required to admit a time dependent parametrisation

$$\gamma^i \in C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T] \times [0,1]) =: \mathbb{E}_T^i \,.$$

As a consequence we have to restrict ourselves to initial networks that are admissible in the sense that they satisfy certain compatibility conditions. Among some other geometric assumptions they need to admit a parametrisation of class $C^{4+\alpha}$ and satisfy all the boundary conditions that need to be valid during the evolution of the respective flow.

Theorem 0.1. *Let $\mathcal{N}_0$ be a geometrically admissible initial network for the C^0–flow (or the C^1–flow) of a Theta Network or the C^0–flow (or C^1–flow) of a Triod with Navier boundary conditions, respectively. Then there exists a positive time T such that within the interval $[0,T]$ the respective flow admits a unique solution $(\mathcal{N}(t))$.*

The solution is unique in the sense of networks as sets. We state two of the four possible boundary value problems that are covered by the above Theorem. In the following we denote by $\nu^i = R\tau^i$ the normal of the respective curve being the counterclockwise rotation of the tangent τ^i by $\frac{\pi}{2}$. The subscripts s and t refer to differentiation with respect to time and the arclength parameter, respectively. All equations are supposed to be valid in the entire time interval $[0,T]$.

Definition 0.2 (C^1–flow for a Theta Network)**.** A family of Theta Networks $(\Theta(t))$ solves the C^1–flow with admissible initial network Θ_0 on $[0,T]$ if there exist parametrisations $\gamma^i \in \mathbb{E}_T^i$ such that for $y \in \{0,1\}$ and $i \in \{1,2,3\}$

$$\begin{cases} \left\langle \gamma_t^i, \nu^i \right\rangle = -2k_{ss}^i - \left(k^i\right)^3 + \mu\, k^i & \text{motion,} \\ \gamma^1\,(t,y) = \gamma^2\,(t,y) = \gamma^3\,(t,y) & \text{concurrency condition,} \\ \tau^1(t,y) + \tau^2(t,y) + \tau^3(t,y) = 0 & \text{angle condition,} \\ k^1(t,y) + k^2(t,y) + k^3(t,y) = 0 & \text{curvature condition,} \\ \sum_{i=1}^3 \left(2k_s^i \nu^i + (k^i)^2 \tau^i\right)(t,y) = 0 & \text{third order condition,} \\ \Theta(0) = \Theta_0 & \text{initial data}\,. \end{cases} \tag{0.1}$$

Definition 0.3 (C^0–flow for a Triod with Navier condition at the fixed endpoints)**.** A family $(\mathbb{T}(t))$ is a solution to the Navier C^0–flow for a Triod with admissible initial network $\mathbb{T}_0$ if there exist parametrisations $\gamma^i \in \mathbb{E}^i_T$ such that

$$\begin{cases} \left\langle \gamma^i_t, \nu^i \right\rangle = -2k^i_{ss} - \left(k^i\right)^3 + \mu\, k^i & \text{motion,} \\ \gamma^1\,(t,0) = \gamma^2\,(t,0) = \gamma^3\,(t,0) & \text{concurrency condition,} \\ k^i(t,0) = 0 & \text{curvature condition,} \\ \sum_{i=1}^3 \left(2k^i_s \nu^i - \mu\tau^i\right)(t,0) = 0 & \text{third order condition,} \\ \gamma^i(t,1) = P^i & \text{fixed endpoints,} \\ k^i(t,1) = 0 & \text{Navier condition,} \\ \mathbb{T}(0) = \mathbb{T}_0 & \text{initial data}\,. \end{cases} \tag{0.2}$$

The strategy to prove existence and uniqueness of classical solutions can be divided into three steps. First we derive a parabolic quasilinear fourth order system of PDEs for the parametrisation which we then solve in a unique way. The last step is to pass from the parametrisations and the PDE back to the networks and the degenerate problem. For details we refer to [3].

In the systems (0.1) and (0.2) only the normal velocity is prescribed. To turn this degenerate equation into a parabolic equation one has to specify a suitable tangential movement which at the 3–points is uniquely determined by the normal velocity and the concurrency constraint. The resulting non-degenerate system for the parametrisation is under-determined in the sense that at each triple junction three scalar boundary conditions are missing. To remove these tangential degrees of freedom at the 3–points one has to carefully choose conditions on the parametrisation that on the one hand yield a well-posed PDE and on the other hand do not affect the geometric problem. The condition

$$\left\langle \gamma^i_{xx}, \tau^i \right\rangle = 0 \qquad \text{at 3–points}$$

satisfies these requirements. To prove existence and uniqueness of the PDE we linearise all equations around a fixed initial parametrisation. The unique solvability of the linear system relies on the fact that the coupled boundary conditions are compatible. Here we use the theory in [4]. We obtain existence and uniqueness of the original PDE with a fixed point argument. To come back to the geometric problem the crucial point is to check that the tangential velocity and the boundary conditions one has chosen don't change the geometry of the problem but can always be obtained by parametrising the objects appropriately. To do this we follow the approach in [2].

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References

[1] J. W. Barrett, H. Garcke, and R. Nrnberg, Math. Models Methods Appl. Sci. **22**, pp. 1250037 – 57.

[2] H. Garcke and A. Novick-Cohen, Adv. Differential Equations **5**, pp. 401 – 434.

[3] H. Garcke, J. Menzel, and A. Pluda, ArXiv e-prints, arXiv:1711.11504 [math.AP].

[4] V. A. Solonnikov, Boundary value problems of mathematical physics. III, Proceedings of the Steklov Institute of Mathematics no. 83 (1965), Amer. Math. Soc., Providence, R.I., 1967.